Bremsstrahlung - free free emission

Asaf Pe'er[1]

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This part of the course is based on Ref. [1].

1. Introduction

As we discussed, Coulomb scattering is natural to fully ionized plasma. The acceleration of charged particles by the Coulomb field of another charged particle leads to the emission of radiation. This radiation is called **Bremsstrahlung** (from German: "breaking radiation"), or **free-free**, as the scattered particle remain free after the collision.

Here, we will derive this radiation using classical (non-relativistic) treatment, and comment on the full quantum-mechanical correction.

We first note that collision of like particles (electron-electron or proton-proton) gives, to zeros order, no radiation, since the dipole moment, $\sum_i q_i \vec{r}_i$ is proportional to the center of mass, $\sum_i m_i \vec{r}_i$ which is a constant of motion. We thus consider electron-ion bremsstrahlung, in which the electrons are the prime source of radiation, since they are accelerated in a much more effective way than the ions. Sine $m_i \gg m_e$, we can treat the ion as fixed in space, while the electron moves in the Coulomb field of the ion.

2. Radiation from an accelerated particle

We recall that when a charged particle is accelerated, it emits radiation in accordance to Larmor's formula,

$$P = \frac{2}{3} \frac{q^2 \dot{u}^2}{(4\pi\epsilon_0)c^3}. \tag{1}$$

Here, p is the total radiated power, and u is the particle's velocity.

This can be generalized to the scenario where we have many non-relativistic particles, $u \ll c$. We can generalize the Larmor formula to write

$$P = \frac{2}{3} \frac{\ddot{\vec{d}}^2}{(4\pi\epsilon_0)c^3}. \tag{2}$$

[1]Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

Here,

$$\vec{d} = \sum_i q_i \vec{r}_i$$

is the **dipole moment** of the radiating particles.

3. Sketch of derivation of Bremsstrahlung emission

The interaction time between the electron and the ion is

$$\tau \approx \frac{b}{v},$$

where b is the impact parameter.

As a crude approximation, we can take the acceleration to be constant during the interaction time. The Coulomb force in the perpendicular direction is

$$F_{C,\perp} = \frac{q^2}{4\pi\epsilon_0 r^2}\cos\theta = \frac{q^2}{4\pi\epsilon_0 b^2}\cos^3\theta \approx \frac{q^2}{4\pi\epsilon_0 b^2} \tag{3}$$

where $\cos\theta = b/r$. Thus,

$$a \approx \frac{q^2}{4\pi\epsilon_0 m_e b^2}$$

Using Larmor's formula (Equation 1), we get

$$P = \frac{2}{3}\frac{q^2 a^2}{(4\pi\epsilon_0)c^3} \approx \frac{q^6}{(4\pi\epsilon_0)^3 m_e^2 b^4 c^3}. \tag{4}$$

The typical frequency is

$$\omega \approx \frac{1}{\tau} = \frac{v}{b},$$

and so the power per unit frequency can be approximated by

$$P(\omega) \approx \frac{P}{\omega} = \frac{q^6}{(4\pi\epsilon_0)^3 m_e^2 v b^3 c^3} \tag{5}$$

We may estimate the impact parameter b using the ions density: $b^3 \approx n_i^{-1}$, or $b \approx n_i^{-1/3}$.

The total **emissivity** (=energy emitted per unit time per unit volume per unit frequency) is obtained by multiplying the power radiated (per unit frequency) from a single electron multiplied by the number density of electrons. If the radiation is isotropic, we need to divide by 4π. We get

$$j(\omega) = \frac{n_e n_i}{4\pi}\frac{q^6}{(4\pi\epsilon_0)^3 m_e^2 c^3}\left(\frac{m_e}{k_B T}\right)^{1/2} \tag{6}$$

where we approximated the electron's velocity with k_BT/m_e.

(In fact, defined this way, j is not the emissivity, but the emission coefficient).

In order to get the total radiated power per unit volume, we need to integrate over the frequencies: $j = \int_0^{\omega_{\max}} j(\omega)d\omega$. We thus need to estimate the maximum frequency, $\omega_{\max}$.

As a first approximation we may assume $\hbar\omega_{\max} = k_BT$, namely that the electron cannot emit a photon with energy higher than the average energy of the electrons. However, clearly, there are electrons with energies higher than the average. On the other hand, their number decreases exponentially with the energy, so we expect that the contribution to the radiation at energies above this $\omega_{\max}$ will be exponentially small. We can thus estimate

$$j = \int_0^{\omega_{\max}} j(\omega)d\omega \simeq \frac{n_e n_i}{4\pi} \frac{q^6}{(4\pi\epsilon_0)^3 m_e^2 c^3} \left(\frac{m_e}{k_BT}\right)^{1/2} \frac{k_BT}{\hbar} = \frac{n_e n_p q^6}{4\pi(4\pi\epsilon_0)^3 m_e^2 c^3} \frac{(m_e k_BT)^{1/2}}{\hbar} \qquad (7)$$

Indeed, an exact calculation gives

$$j(\nu) = \frac{32\pi}{3} \frac{n_e n_i q^6}{(4\pi\epsilon_0)^3 m_e^2 c^3} \left(\frac{2\pi m_e}{3k_BT}\right)^{1/2} e^{-\frac{h\nu}{k_BT}} \bar{g}_{ff}, \qquad (8)$$

and

$$j = \frac{16}{3} \frac{n_e n_i q^6}{(4\pi\epsilon_0)^3 \hbar m_e^2 c^3} \left(\frac{2\pi m_e k_BT}{3}\right)^{1/2} \langle \bar{g}_{ff} \rangle \qquad (9)$$

$\bar{g}_{ff}(\nu, T)$ is a number of order unity, known as the **Gaunt factor**. It provides the QM corrections to the classical formula. $\langle \bar{g}_{ff}(T) \rangle$ is the frequency averaged Gaunt factor.

From Equation 7, we see that the Bremsstrahlung emission ha a flat spectrum, namely equal amount of energy is released per frequency bin.

4. Thermal free free absorption

When we have a thermal (Maxwellian) distribution of particle's energies, we can use Kirchoff's law to find the absorption coefficient:

$$S_\nu = \frac{j_\nu}{\alpha_\nu}. \qquad (10)$$

Here, S_ν is the **source function** of the black body (thermal) radiation,

$$S_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{k_BT}} - 1}$$

$j_\nu \equiv j(\nu)$ is the emission coefficient, and α_ν is the absorption coefficient.

From Kirchoff's law, we get

$$\alpha_\nu^{ff} = \frac{j_\nu}{S_\nu} = \frac{8}{3}\frac{n_e n_i q^6}{(4\pi\epsilon_0)^3 \hbar m_e^2 c^2}\left(\frac{2\pi m_e c^2}{3k_B T}\right)^{1/2}\frac{1-e^{-\frac{h\nu}{k_B T}}}{\nu^3}\bar{g}_{ff} \tag{11}$$

For very low frequencies (the Rayleigh-Jeans regime), $h\nu \ll k_B T$, a good approximation is

$$\alpha_\nu^{ff} \approx 0.018\frac{n_e n_i}{T^{3/2}\nu^2}\bar{g}_{ff} \tag{12}$$

(in cgs units).

This means that the absorption coefficient increases as the frequency decrease. So, for small enough frequency, the optical depth $\tau = \alpha_\nu R$ will become unity (R is a characteristic size of the plasma).

At frequencies lower than that, the radiation becomes self-absorbed: photons are absorbed at a rate faster than they are created. The resulting spectra is that of a black-body, in the Rayleigh-Jeans regime, $I_\nu \propto \nu^2$.

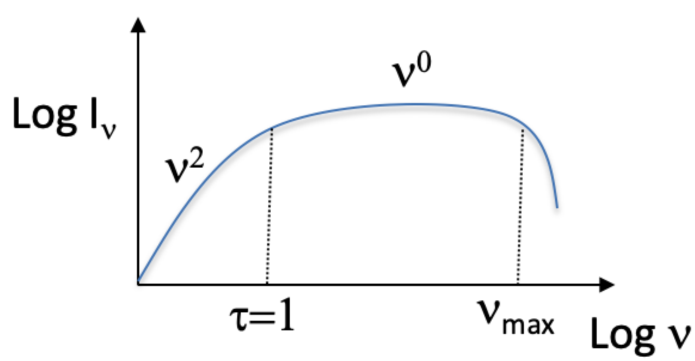

Fig. 1.— Typical spectrum expected from Bremsstrahlung process in plasma.

A. Derivation of Larmor's formula

Maxwell's equations imply that all classical electromagnetic radiation is ultimately generated by accelerating electrical charges. It is possible to derive the intensity and angular distribution of the radiation from a point charge (a charged particle) subject to an arbitrary but small acceleration $\Delta v/\Delta t$ via Maxwell's equations (and retarded potentials), but the complicated math obscures the physical interpretation that remains clear in J. J. Thomson's simpler derivation.

If a particle with electrical charge q is at rest (or moves at aa constant velocity), Coulomb's law implies that its electric field lines will be purely radial: $\vec{E} = E_r$. Suppose that a charged particle initially at rest is accelerated to some small velocity $\Delta v \ll c$ in some short time Δt. This disturbs the lines of force, and the disturbance travels outward at the speed of light c (see Figure 2).

At time t later, the disturbance will have propagated to $r = ct$, and, within the disturbance, there will be a perpendicular component of electric field, with magnitude (relative to the radial component):

$$\frac{E_\perp}{E_r} = \frac{\Delta v t \sin\theta}{c\Delta t} \tag{A1}$$

where θ is the angle between the acceleration vector and the line from the charge to the observer (see Figure 2).

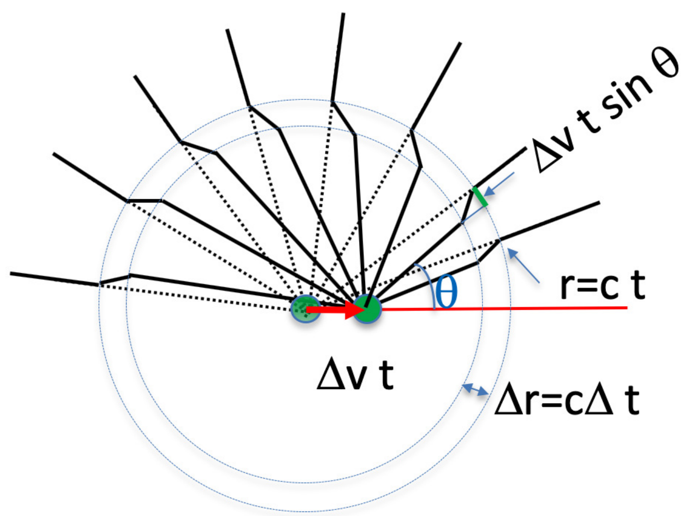

Fig. 2.— Electric field from an accelerated electron.

Coulomb's law for the radial component E_r of the electric field (electric force per unit charge) at a distance r from a charge q (in cgs units) is

$$E_r = \frac{q}{r^2}. \tag{A2}$$

Substituting $t = r/c$ gives

$$E_\perp = \frac{q}{r^2}\left(\frac{\Delta v}{\Delta t}\right)\frac{r\sin\theta}{c^2} \tag{A3}$$

or, taking the limit $\Delta t \to 0$,

$$E_\perp = \frac{q\dot{v}\sin\theta}{rc^2} \tag{A4}$$

This equation is valid for any small acceleration. The transverse field $E_\perp$ instantaneously reflects the applied acceleration. Note that $E_\perp \propto r^{-1}$, while $E_r \propto r^{-2}$. This means that far from the charges particle - namely, at large r, only $E_\perp$ will have a asignificant contribution to the radiation field.

Furthermore, note that from the observer's point of view, only the visible acceleration perpendicular to the line of sight ($\dot{v}\sin\theta$) contributes to the radiated electric field; the invisible component of acceleration parallel to the line of sight does not radiate

The radiated power is given by the Poynting flux = the power per unit area (with units $[erg/cm^2/s]$ in cgs unit system). It is

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B}, \tag{A5}$$

where in cgs units $|\vec{B}| = |\vec{E}|$, and so

$$|\vec{S}| = \frac{c}{4\pi}E^2 \tag{A6}$$

Thus, we get

$$|\vec{S}| = \frac{c}{4\pi}\left(\frac{q\dot{v}\sin\theta}{rc^2}\right)^2 = \frac{1}{4\pi}\frac{q^2\dot{v}^2}{c^3}\frac{\sin^2\theta}{r^2} \tag{A7}$$

Thus, the charge radiated with a dipolar power, that looks like a doughnut whose axis is parallel to $\dot{v}$.

The total power emitted by the particle is obtained by integrating over all directions:

$$P = \int_{sphere}|\vec{S}|dA$$

using $dA = r^2\sin\theta d\theta d\phi$, we get

$$P = \frac{q^2\dot{v}^2}{4\pi c^3}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\frac{\sin^2\theta}{r^2}r^2\sin\theta = \frac{q^2\dot{v}^2}{2c^3}\int_0^{\pi}\sin^3\theta d\theta \tag{A8}$$

The integral $\int_0^\pi \sin^3\theta d\theta = 4/3$, and so the total emitted power is

$$P = \frac{2}{3}\frac{q^2\dot{v}^2}{c^3}. \tag{A9}$$

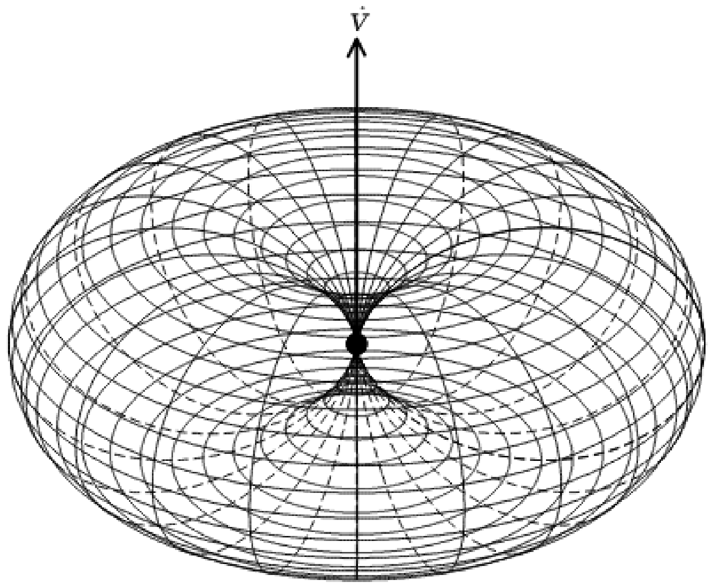

Fig. 3.— The power pattern of Larmor radiation from a charged particle. The power received in any direction is proportional to the component of $\dot{v}$ perpendicular to the line of sight. Figure taken from Ref. [3]

This result is known as **Larmor's equation**, and appears in Equation 1.

Larmor's equation states that any charged particle radiates when accelerated and that the total radiated power is proportional to the square of the acceleration. For an electromagnetic acceleration the acceleration is usually proportional to the charge/mass ratio of the particle. Since $m_p/m_e = 1836 \approx 2000$ while they have the same charge, we typically expect radiation from electrons to be $\approx 4 \times 10^6$ stronger than radiation from protons.

REFERENCES

[1] G. Rybicki & A. Lightman, *Radiative Processes in Astrophysics*

[2] M. Longair, *High Energy Astrophysics.*

[3] Lecture notes by Jim Condon (NRAO) on radio astronomy, see
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