Single particle motion

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December 11, 2019

This part of the course is based on Refs. [1], [2] and [3].

1. Introduction

We now start a quantitative discussion of plasma properties. The first difficulty, is that the densities in which plasmas can be found span an enormous range of many orders of magnitude. The simplest case is a one in which the densities are low enough, such that we can ignore collective (fluid-like) behavior, and look at trajectories of single particles.

When we discuss *single particle motion*, it means that **we ignore the influence of the particle current on the electric and magnetic fields**. While this is of course incomplete, we can gain a lot of useful insight.

2. Motion in static electric and magnetic fields

Newton's equation of motion for a particle of mass m and charge q that is subject to electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ is

$$m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right). \tag{1}$$

Though simple, analytical solutions exist only for simple cases, such as homogeneous and stationary fields.

2.1. $\mathbf{E} = 0$ and the cyclotron frequency

Let us choose $\mathbf{E} = 0$, and magnetic field along the $\hat{z}$ direction, $\mathbf{B} = B\hat{z}$. The equation of motion (1) becomes

$$\begin{array}{c} m\frac{dv_x}{dt} = qBv_y\ , \quad m\frac{dv_y}{dt} = -qBv_x\ , \quad m\frac{dv_y}{dt} = 0 \\ \frac{d^2v_x}{dt^2} = \frac{qB}{m}\frac{dv_y}{dt} = -\left(\frac{qB}{m}\right)^2 v_x, \\ \frac{d^2v_y}{dt^2} = -\frac{qB}{m}\frac{dv_x}{dt} = -\left(\frac{qB}{m}\right)^2 v_y. \end{array} \tag{2}$$

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These are the equations of a simple harmonic oscillator, which describes a periodic motion at a frequency

$$\boxed{\omega_c = \frac{|q|B}{m},} \tag{3}$$

which is known as the **cyclotron frequency**.

A formal solution to the Equation of motion 2 is then

$$v_{x,y} = v_\perp \exp\left(\pm i\omega_c t + i\delta_{x,y}\right), \tag{4}$$

where $v_\perp$ is the velocity perpendicular to the magnetic field (i.e., in the $x - y$ plane), the $\pm$ sign denotes the sign of q, and $\delta_{x,y}$ is the phase.

We can choose the phase δ such that

$$v_x = v_\perp e^{i\omega_c t}, \tag{5}$$

and then

$$v_y = \frac{m}{qB}\frac{dv_x}{dt} = \pm\frac{1}{\omega_c}\frac{dv_x}{dt} = \pm i v_\perp e^{i\omega_c t}. \tag{6}$$

Integrating once again, we get

$$x - x_0 = -i\frac{v_\perp}{\omega_c}e^{i\omega_c t} \quad , \quad y - y_0 = \pm\frac{v_\perp}{\omega_c}e^{i\omega_c t}. \tag{7}$$

We can now define the **Larmor radius**,

$$\boxed{r_L \equiv \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|B},} \tag{8}$$

from which (taking the real part) we write Equation 7 as

$$x - x_0 = r_L \sin(\omega_c t) \quad , \quad y - y_0 = \pm r_L \cos(\omega_c t). \tag{9}$$

These describe a circular orbit about a fixed *guiding center* (x_0, y_0) - see Figure 1.

2.2. Finite E and drift

Next, we consider motion under combined influence of **B** and **E** fields. Without loss of generality, we can choose **E** lie in the $x - z$ plane.

Decomposing the equation of motion (1) into its components, we get in the z direction:

$$\frac{dv_z}{dt} = \frac{q}{m}E_z \quad \rightarrow \quad v_z = \frac{qE_z}{m}t + v_{z,0}, \tag{10}$$

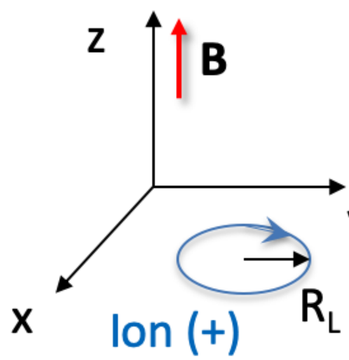

Fig. 1.— Larmor orbit of an ion in a magnetic field

namely a linear acceleration along the direction of **B**.

As for the transverse components, we get

$$\begin{aligned} \frac{dv_x}{dt} &= \frac{q}{m}E_x \pm \omega_c v_y, \\ \frac{dv_y}{dt} &= 0 \mp \omega_c v_x. \end{aligned} \tag{11}$$

Differentiating, we get (for constant **E**),

$$\begin{aligned} \frac{d^2v_x}{dt^2} &= -\omega_c^2 v_x, \\ \frac{d^2v_y}{dt^2} &= \mp\omega_c \left(\frac{q}{m}E_x \pm \omega_c v_y\right) = -\omega_c^2 \left(\frac{E_x}{B} + v_y\right). \end{aligned} \tag{12}$$

The last equation can be written as

$$\frac{d^2}{dt^2}\left(v_y + \frac{E_x}{B}\right) = -\omega_c^2 \left(v_y + \frac{E_x}{B}\right).$$

Comparing with Equation 2 for $\mathbf{E} = 0$, we find that we retrieve the same equation if we replace $v_y \to v_y + (E_x/B)$.

The solution is therefore similar to equations 4, 5,

$$\begin{aligned} v_x &= v_\perp e^{i\omega_c t}, \\ v_y &= \pm i v_\perp e^{i\omega_c t} - \frac{E_x}{B}. \end{aligned} \tag{13}$$

We thus find that the Larmor radius is the same as in the $\mathbf{E} = 0$ case, but there is a superimposed drift velocity $\mathbf{v}_g$ of the guiding center in the $-y$ direction (for $E_x > 0$). This is illustrated in Figure 2. Note that while the Larmor radius of electrons is obviously much

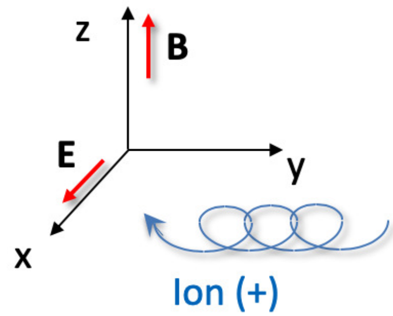

Fig. 2.— Drift motion of an ion in an EM field. Here, the magnetic field is along the $\hat{z}$ direction, $\mathbf{B} = B\hat{z}$ and the electric field is in the $\hat{x}$ direction, $\mathbf{E} = E\hat{x}$. Hence, the particle moves in the $x - y$ plane, drifting along the $-y$ direction.

smaller than that of the ions (due to the much smaller mass), the direction of the drift velocity is the same for both.

If we were to carry a vectorial form of the equations all the way, we would obtain the electric field drift of the guiding center as

$$\mathbf{v}_E \equiv \mathbf{v}_{\perp,g} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}, \tag{14}$$

where $\mathbf{v}_E$ is the electric field drift of the guiding center.

This can be seen be using $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ (the $md\mathbf{v}/dt$ term is responsible only to the circular motion), then taking the cross product with $\mathbf{B}$, to get

$$\mathbf{E} \times \mathbf{B} = \mathbf{B} \times (\mathbf{v} \times \mathbf{B}) = \mathbf{v}B^2 - \mathbf{B}(\mathbf{v} \cdot \mathbf{B})$$

from which the transverse component of the velocity is readily found.

2.3. Gravitational drift

When considering ionospheric plasma at the magnetic equator, we encounter a similar situation to the crossed electric and magnetic fields. We only need to replace the electric force, $q\mathbf{E}$ with the gravitational force, $m\mathbf{g}$ (or, really, any other force that may be). The guiding center drift caused by the gravitational force is thus

$$\mathbf{v}_g = \frac{m}{q} \frac{\mathbf{g} \times \mathbf{B}}{B^2} \tag{15}$$

Note that the drift velocity due to gravity depends on both the mass and the charge. Thus, as opposed to drift due to pure EM forces, when the cause of the drift is the gravitational force, electrons and ions drift *in opposite directions*. The gravitational drift is responsible for an equatorial net electric current that is driven by the weight force on the plasma.

3. Nonuniform magnetic field

Let us now consider the motion of particles under the influence of varying **E** and **B** fields. The variation can be either in space or in time (or both). In general, there is no simple analytical solution to the problem. What we will do, is to expand in the small ratio r_L/L, where L *is the scale length of the inhomogeneity.* This approach is known as **orbit theory**.

3.1. Grad-B drift: $\nabla\mathbf{B} \perp \mathbf{B}$.

The assumption that the change of the magnetic field is small over one gyro-radius (with respect to the magnitude of the magnetic field) can be written (for $\mathbf{B} = B\hat{z}$) as

$$r_L \frac{\partial B_z}{\partial r} \ll B_z. \tag{16}$$

We assume that the magnetic field lines are straight, but that their density increases-say, in the y direction (see Figure 3). Physically, the gradient causes a decrease in the Larmor radius with y, and we thus expect a drift, perpendicular to both **B** and ∇**B**. The direction of the drift should be opposite to electrons and ions.

Consider the Lorentz force, $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, averaged over a gyration. In the geometry shown in Figure 3, the magnetic field only varies with $\hat{y}$, namely $\mathbf{B} = B_z(y)$. Thus, the net force in the $\hat{x}$ direction averages out.

The net force in the $\hat{y}$ direction is

$$F_y = -qv_x B_z(y) \tag{17}$$

where $B_z(y)$ is the magnetic field in the position of the particle. We now use Taylor expansion, $B_z(y) = B_0 + y(t)(\partial B_z/\partial y)$, which is valid as long as $r_L \ll L$, where L is the typical scale length of change of $\partial B_z/\partial y$. With the help of Equations 5, 9, we write

$$F_y = -qv_\perp \cos(\omega_c t) \left[B_0 \pm r_L \cos(\omega_c t) \frac{\partial B_z}{\partial y} \right]. \tag{18}$$

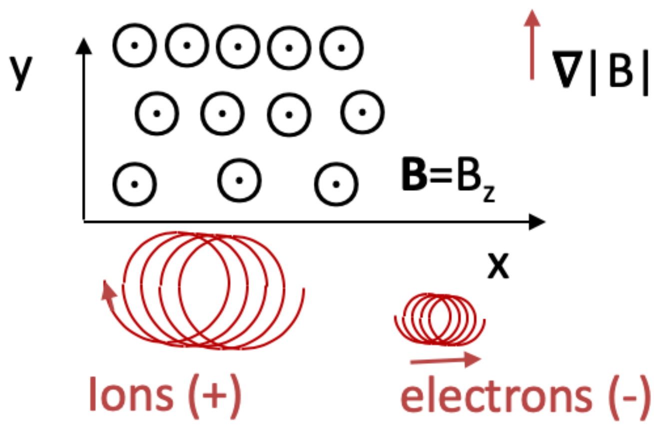

Fig. 3.— The drift of a gyrating particle in a nonuniform magnetic field.

We can now average over one gyration period. The first term in Equation 18 averages to 0, while the average of $\cos^2(\omega_c t)$ is 1/2. Thus,

$$\langle F_y \rangle = \mp q v_\perp \frac{r_L}{2} \frac{\partial B_z}{\partial y} \tag{19}$$

(The upper sign for positive charge, the lower one for negative).

The drift velocity of the guiding center is thus

$$\mathbf{v}_{\nabla B} = \frac{1}{q} \frac{\langle \mathbf{F} \rangle \times \mathbf{B}}{B^2} = \mp \frac{1}{2} \frac{v_\perp r_L}{B_z} \frac{\partial B_z}{\partial y} \hat{x} \tag{20}$$

In a general vectorial form, we get

$$\mathbf{v}_{\nabla B} = \pm \frac{1}{2} v_\perp r_L \frac{\mathbf{B} \times \nabla \mathbf{B}}{B^2}. \tag{21}$$

The quantity $\mathbf{v}_{\nabla B}$ is known as the **gradient drift**. The charge dependence leads to charge separation, and to the formation of a net current.

3.2. Curvature drift

We next consider a situation where the field lines are curved with a constant radius of curvature, R_c (see Figure 4). We neglect the gradient of the magnetic field, namely, we assume $|B|$ to be constant [2].

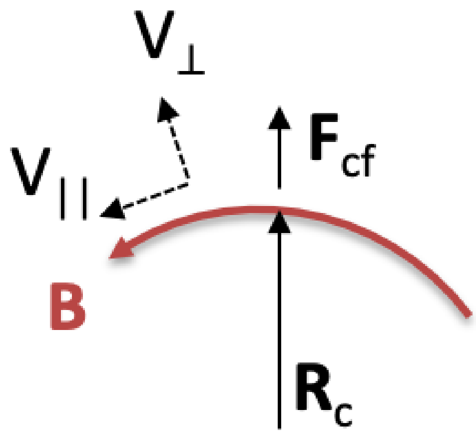

Fig. 4.— A curved magnetic field

Denoting by $v_\parallel$ the component of the velocity parallel to the magnetic field, the average centrifugal force (over a gyro-period) is

$$\langle \mathbf{F}_{cf} \rangle = \frac{mv_\parallel^2}{R_c}\hat{r} = mv_\parallel^2 \frac{\mathbf{R}_c}{R_c^2}. \tag{22}$$

Using this force in the general Equation for the drift velocity (Equations 14, 15), we get the **curvature drift velocity**,

$$\mathbf{v}_R = \frac{1}{q}\frac{\langle \mathbf{F}_{cf} \rangle \times \mathbf{B}}{B^2} = \frac{mv_\parallel^2}{qB^2}\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2} \tag{23}$$

3.2.1. The toroidal drift

So far, we have neglected the gradient of **B**. However, if the magnetic field is generated by currents that flow outside of the volume of interest, there is also a gradient in the magnetic field, hence we must add the gradient drift to the result obtained in Equation 23.

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[2]Such a field does not obey Maxwell's equations in vacuum, so a $\nabla|B|$ field will always be present, but we already discussed it earlier.

In vacuum, $\nabla \times \mathbf{B} = 0$. In the cylindrical coordinates $(\hat{r}, \hat{\theta}, \hat{z})$ used in Figure 4, $\mathbf{B}$ has only a $\hat{\theta}$ component, while $\nabla\mathbf{B}$ has only $\hat{r}$ component. Hence, $\nabla \times \mathbf{B}$ has only a z component,

$$(\nabla \times \mathbf{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 0$$

(in vacuum). Thus,

$$|B| \propto B_\theta \propto \frac{1}{R_c}\ , \quad \frac{\nabla|B|}{|B|} = -\frac{\mathbf{R}_c}{R_c^2} \tag{24}$$

Using Equation 21 for the drift velocity, we thus find

$$\mathbf{v}_{\nabla B} = \mp\frac{1}{2}\frac{v_\perp r_L}{B^2}\mathbf{B} \times |B|\frac{\mathbf{R}_c}{R_c^2} = \frac{1}{2}\frac{m}{q}v_\perp^2\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2} \tag{25}$$

where in the last equality we used the definition of r_L in Equation 8, $r_L = mv_\perp/|q|B$.

Adding the result in Equation 25 to the curvature drift in Equation 23, we get

$$\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{m}{q}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2}. \tag{26}$$

This sum is often called the **toroidal drift** because of its role in magnetic confinement in torus-like configurations. Unfortunately, as we can see, the drifts add. This means that if one bends a magnetic field into a torus for the purpose of confining a plasma, it will always drift out of the torus.

For a Maxwellian distribution, $\langle v_\parallel^2 \rangle$ and $\frac{1}{2}\langle v_\perp^2 \rangle$ are both equal to $k_B T/m$ (note that $v_\perp$ has 2 degrees of freedom). The average curved-field drift is then

$$\langle \mathbf{v}_{R+\nabla B} \rangle = \pm\frac{mv_{th}^2}{qBR_c} = \pm\frac{\langle r_L \rangle}{R_c}v_{th}\hat{z} \tag{27}$$

where $\hat{z}$ is the direction of $\mathbf{R}_c \times \mathbf{B}$, and $v_{th} = \sqrt{2k_B T/m}$ is the thermal velocity of the particles. This drift velocity thus depends on the charge, but not the mass of the particles.

3.3. Longitudinal gradient $\nabla\mathbf{B}||\mathbf{B}$

We now consider a magnetic field $\mathbf{B} = B\hat{z}$, whose magnitude also varies in the $\hat{z}$ direction (see Figure 5). We assume axisymmetric field, $B_\theta = 0$, $\frac{\partial B_z}{\partial \theta} = 0$. Since the magnetic field lines converge / diverge, there is an inevitable radial component, B_r. This produces a force that can trap particles inside the magnetized region.

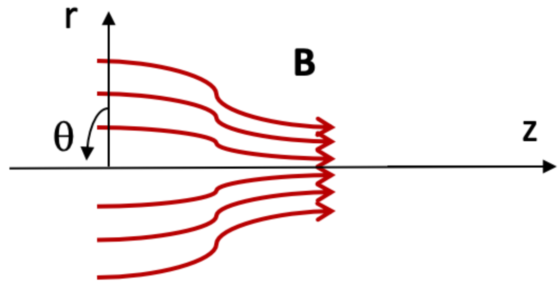

Fig. 5.— A magnetic mirror configuration

We expand $\nabla \cdot \mathbf{B} = 0$ to calculate B_r:

$$\frac{1}{r}\frac{\partial (rB_r)}{\partial r} + \frac{\partial B_z}{\partial z} = 0. \tag{28}$$

If we know $\frac{\partial B_z}{\partial z}$ at $r = 0$, and assume it does not vary much along r, we can calculate

$$\begin{array}{l} rB_r = -\int_0^r r\frac{\partial B_z}{\partial z}dr \simeq -\frac{r^2}{2}\left[\frac{\partial B_z}{\partial z}\right]_{r=0}, \\ \rightarrow \quad B_r \simeq -\frac{r}{2}\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \end{array} \tag{29}$$

Since $B_\theta = 0$, we find that the Lorentz force in the $\hat{z}$ direction is

$$F_z = q(v_r B_\theta - v_\theta B_r) = -qv_\theta B_r \simeq \frac{1}{2}qv_\theta r\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \tag{30}$$

We now average over one gyrotime. For a particle whose guiding center lies on the $\hat{z}$ axis, $v_\theta = \mp v_\perp$ (depending on the sign of q), and $r = r_L$. Thus, the average force is

$$\langle F_z \rangle = \mp\frac{1}{2}qv_\perp r_L\left[\frac{\partial B_z}{\partial z}\right]_{r=0} = -\frac{1}{2}\frac{mv_\perp^2}{B}\left[\frac{\partial B_z}{\partial z}\right]_{r=0}. \tag{31}$$

3.3.1. Magnetic moment

The gyrating motion of the particle about the central field line can be viewed as an electric current ring. Such a ring carries a current which is the rate of flow of an electric charge, $I = qv_\perp/(2\pi r_L)$. The area of the ring is $A = \pi r_L^2$, hence the magnetic moment is

$$\boxed{\mu = IA = \frac{1}{2}qv_\perp r_L = \frac{1}{2}\frac{mv_\perp^2}{B}.} \tag{32}$$

We can therefore write the average force (Equation 31) as

$$\langle F_z \rangle = -\mu\frac{\partial B_z}{\partial z} \tag{33}$$

4. Adiabatic Invariants

In classical mechanics, the action integral over a periodic orbit, $\oint pdq$ is a conserved quantity of the system - a constant of motion. Here, p and q are the generalized momentum and coordinates. This concept can be extended when weak gradients exist. By "weak" we mean that the changes are slow compared to the period of motion, such that the action integral $\oint pdq$ is still well defined, although formally this is no longer an integral over a closed path. Then, the orbits are *nearly* periodic. The associated action integrals are no longer strict invariants, but are called **adiabatic invariants**.

4.1. The first adiabatic invariant: the magnetic moment

As a particle moves into regions of stronger or weaker B, its Larmor radius changes, but the magnetic moment, μ, is invariant. To prove that, start from Equation 33:

$$m\frac{dv_\parallel}{dt} = -\mu\frac{\partial B}{\partial z}. \tag{34}$$

We multiply by $v_\parallel$ on the left, and, equivalently, dz/dt on the right,

$$mv_\parallel\frac{dv_\parallel}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\frac{\partial B}{\partial z}\frac{dz}{dt} = -\mu\frac{dB}{dt} \tag{35}$$

Note that dB/dt is the variation of B as is experienced by the guiding center along the particle's motion, while B itself is fixed in time.

A time-invariant (or time-independent) magnetic field does not alter the particle's kinetic energy, as $\mathbf{F}\cdot d\mathbf{s} = q(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}dt = 0$, because the Lorentz force is always perpendicular to the trajectory. We write the conservation of the particle's energy as

$$0 = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right) = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2 + B\mu\right) \tag{36}$$

where in the last equality we used Equation 32.

Combining Equations 35 and 36, we get

$$\begin{aligned} &-\mu\tfrac{dB}{dt} + \tfrac{d}{dt}(\mu B) = 0 \\ &\Rightarrow \tfrac{d\mu}{dt} = 0. \end{aligned} \tag{37}$$

4.1.1. Magnetic mirror

The invariance of μ is the basis for one of the primary schemes for plasma confinements, that of **magnetic mirror**. As a particle moves from a region of weak B to a region of strong B, as B increases, $v_\perp$ must also increase such that μ is constant (Equation 32). Since the particle's total energy is constant, $v_\parallel$ must then decrease. If B is high enough, eventually $v_\parallel$ becomes 0, and the particle is "reflected" back to the weak field region.

We note, though, that the mirror is not perfect. If the particle has initially (when subject to the weak field, B_0) a parallel velocity component, we may write $v_{\perp,0} = v_0 \sin\theta$. In this case, the reflection occurs when $v_\parallel = 0$, namely

$$0 = v_\parallel^2 = v_0^2 \left(1 - \frac{B}{B_0}\sin^2\theta\right),$$

where we used the fact that $v_\perp^2/B = v_{\perp,0}^2/B_0$. Here, B is the magnetic field in the reflection point.

This result implies that the reflection point does not depend on the particle's initial velocity, but does depend on its initial angle: all particles with a starting angle $\theta > \theta_m$ are confined, while particles with $\theta < \theta_m$ (here, θ_m is a characteristic angle) will overcome the mirror point and thus form a **loss cone** in velocity space. If we denote the magnetic field at the mirror point by B_m, then $\sin\theta_m = \sqrt{B_0/B_m}$.

4.2. The longitudinal invariant

Consider a particle trapped between 2 magnetic mirrors. It bounces between them, and therefore have a periodic motion at the "bounce frequency". A constant of this motion is therefore $\oint m v_\parallel ds$, where ds is an element of path (of the guiding center) along a field line (see Figure 6).

Since the guiding center itself drifts across field lines, the the motion is not exactly periodic, and therefore the constant of motion becomes an adiabatic invariant, which is called the **longitudinal invariant**, J which is defined for a half-cycle between 2 turning points, a and b,

$$J = \int_a^b v_\parallel ds. \tag{38}$$

A good example of the usefulness of J is the earth's magnetic field (see Figure 6). Being a dipole, this field drops as $1/r^3$, hence it mirror-traps charged particles which bounce between the north and south magnetic poles.

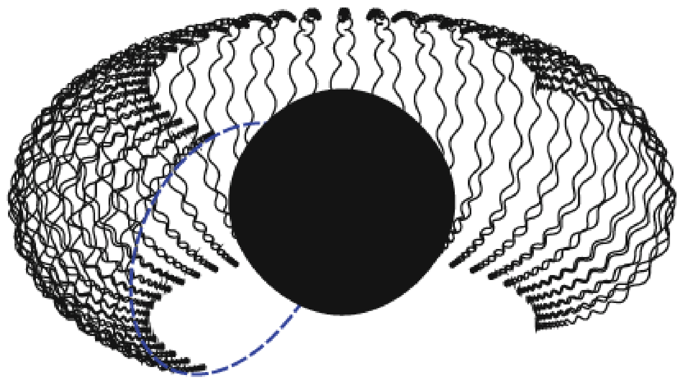

Fig. 3.12 Mirror effect and particle drifts in the Earth's dipole field. A 10 MeV proton with a pitch angle of $\theta = 30°$ starting at $3R_E$ is trapped in the Earth magnetic field. The initial field line, on which the particle motion started, is shown by the *dashed curve*. The particle performs a hierarchy of three periodic motions: gyration about the field line, bouncing between the mirror points, and a slow (toroidal) drift in the equatorial plane

Fig. 6.— Particle motion in the earth's magnetic dipole field: mirror effect and particle drift. Figure taken from Ref. [2]

Due to the drift, the particles slowly change their longitude along the earth. After a while, the particle eventually drifts back to the same longitude it started from. Conservation of J (which essentially determines the length of the magnetic field lines between the 2 turning points) ensures that the particle will return to the same line, as no two magnetic field lines have the same length, due to various effects, such as solar wind, etc.

4.3. The third adiabatic invariant: Φ

Referring back to Figure 6, we see a third periodic motion, with the longest period: that associated with the toroidal drift of the guiding center around the earth.

The adiabatic invariant connected to that is the total magnetic flux, Φ, enclosed by the drift surface. Because the drift motion is very long, strong fluctuations in $\mathbf{B}$ occurs often on much shorter time scale, hence the invariance of Φ has very limited practical applicability.

5. Time varying fields

So far, we considered only stationary fields. Let us now look at a few cases of time-varying fields.

5.1. Time varying E field: the polarization drift

Let us consider a case in which $\mathbf{E} = (E_x(t), 0, 0)$ varies in time,

$$\mathbf{E} = E_0 e^{i\omega t}\hat{x}, \tag{39}$$

and the magnetic field $\mathbf{B} = B_0\hat{z}$ is fixed.

The equation of motion is still given by Equation 1, $md\mathbf{v}/dt = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, or

$$\dot{v}_x = \frac{q}{m}E_x + \frac{qB}{m}v_y \ , \quad \dot{v}_y = -\frac{qB}{m}v_x.$$

Taking the derivative, we get

$$\begin{aligned} \ddot{v}_x &= -\omega_c^2 v_x \pm \omega_c \tfrac{\dot{E}_x}{B}, \\ \ddot{v}_y &= -\omega_c^2 v_y - \omega_c^2 \tfrac{E_x}{B}. \end{aligned} \tag{40}$$

Using now $\dot{E}_x = i\omega E_x$, we get

$$\ddot{v}_x = -\omega_c^2 \left(v_x \mp \frac{i\omega}{\omega_c}\frac{E_x}{B} \right) = -\omega_c^2 \left(v_x \mp \frac{1}{\omega_c}\frac{\dot{E}_x}{B} \right) \tag{41}$$

(the upper sign corresponds to positive ions, the lower to negative electrons).

We may now define the **polarization drift** in the direction of the electric field,

$$\boxed{\mathbf{v}_p = \pm\frac{1}{\omega_c}\frac{\dot{E}_x}{B}.} \tag{42}$$

There is also the time dependent $\mathbf{E} \times \mathbf{B}$ drift in the $\hat{y}$ direction,

$$\mathbf{v}_E = -\frac{E_x(t)}{B} \tag{43}$$

which enable us to write Equations 40 as

$$\begin{aligned} \ddot{v}_x &= -\omega_c^2 \left(v_x - v_p \right), \\ \ddot{v}_y &= -\omega_c^2 \left(v_y - v_E \right). \end{aligned} \tag{44}$$

We note that a solution as a sum of the gyromotion and the drift motion of the form

$$v_x = v_\perp e^{i\omega_c t} + v_p,$$
$$v_y = \pm i v_\perp e^{i\omega_c t} + v_E$$

would result in

$$\ddot{v}_x = -\omega_c^2 v_x + (\omega_c^2 - \omega^2) v_p,$$
$$\ddot{v}_y = -\omega_c^2 v_y + (\omega_c^2 - \omega^2) v_E.$$

This solution is **not** identical to Equation 44, unless $\omega^2 \ll \omega_c^2$, namely a very slowly varying **E**.

Physically, the polarization drift can be considered as a switch-on effect of the plasma, which reflects the inertia of the particles. When the electric field is suddenly switch on at $t = 0$, a particle that is initially at rest starts moving in the direction of the field, until the Lorentz force bends its trajectory to the perpendicular direction (see Figure 7). Note that the effect has **opposite signs** for electrons and ions, which leads to a net displacement of of the charge, resulting in a net polarization of the plasma - hence its name.

Fig. 3.13 **a** Polarization by the sudden switch-on of an electric field perpendicular to a static magnetic field. The ion trajectory is on average displaced by a gyroradius. **b** Polarization drift of positive ions in a linearly increasing electric field and a perpendicular magnetic field. Note that in both diagrams the scaling in y-direction is compressed with respect to the x-scaling. The motion of the guiding center is indicated by *dotted lines*

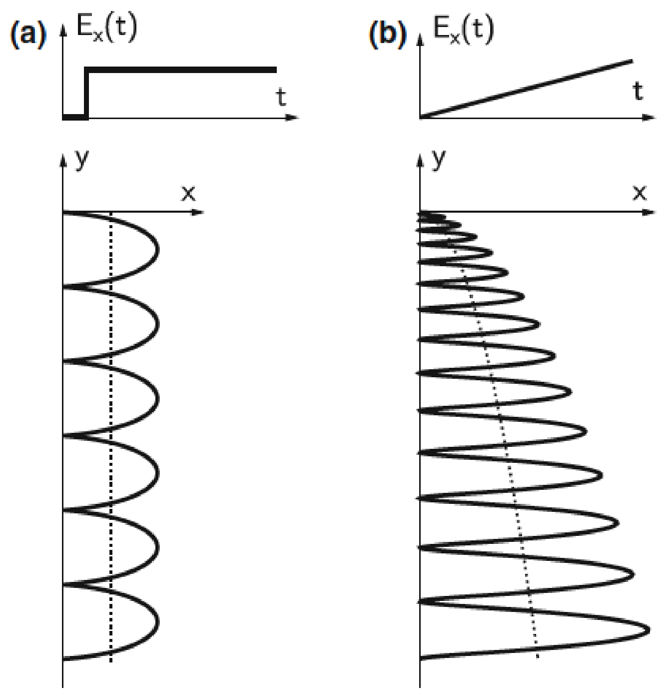

Fig. 7.— Polarization drift for sudden switch-on of electric field, and gradual increase. Figure taken from Ref. [2]

5.2. Time varying B field: conservation of the magnetic moment.

Consider a gyrating particle with Larmor radius, $r_L = v_\perp/\omega_c$ orbiting in a homogeneous magnetic field $\mathbf{B}(t)$. Let us assume that the magnetic field is slowly increasing.

The change in the magnetic field induces an electric field, according to Faraday's law of induction,

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \tag{45}$$

In order to proceed, we neglect the velocity parallel to the magnetic field, and assume that $\mathbf{v} = v_\perp$. Then we take the dot product of the Equation of motion (1) with $v_\perp$, to write

$$m\frac{dv_\perp}{dt} \cdot v_\perp = \frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = q\mathbf{E} \cdot \mathbf{v}_\perp = q\mathbf{E} \cdot \frac{d\mathbf{l}}{dt}, \tag{46}$$

where $\mathbf{l}$ is a unit length along the particle's trajectory.

We now have to carry the integral over time. However, for slowly varying field we can replace the time integral by a line integral of the unperturbed orbit. Over one gyrotime we get

$$\begin{aligned} \delta\left(\tfrac{1}{2}mv_\perp^2\right) &= \oint q\mathbf{E} \cdot \tfrac{d\mathbf{l}}{dt}dt = \oint q\mathbf{E} \cdot d\mathbf{l} \\ &= q\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -q\int_S \dot{\mathbf{B}} \cdot d\mathbf{S} \end{aligned} \tag{47}$$

Here, $\mathbf{S}$ is the surface enclosed by the Larmor orbit of the particle.

We further have $\dot{\mathbf{B}} \cdot d\mathbf{S} < 0$ for ions and > 0 for electrons. We can thus write

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \pm q\frac{dB}{dt}\pi r_L^2 = \left(\frac{\frac{1}{2}mv_\perp^2}{B}\right)\left(\frac{2\pi\frac{dB}{dt}}{\omega_c}\right) \tag{48}$$

where we made use of $r_L = v_\perp/\omega_c$ and $\omega_c = |q|B/m$. The first term on the right hand side is the magnetic moment, μ. The second term is the change of the magnetic field during one period of gyration, which is δB. We thus proved that

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \mu\delta B. \tag{49}$$

Finally, from the definition of μ (Equation 32) we can write $\delta\left(\frac{1}{2}mv_\perp^2\right) = \delta(\mu B)$. This implies that $\delta\mu = 0$, namely that **the magnetic moment is invariant in slowly varying magnetic field**.

As $\mathbf{B}$ changes, the Larmor orbits expand and contract, and the particle loses and gains kinetic energy. However, **the magnetic flux through a Larmor orbit is constant.** This can easily be seen: the magnetic flux through a loop of area S is $BS = B\pi r_L^2$. Thus,

$$\Phi = B\pi r_L^2 = \frac{\pi m^2 v_\perp^2}{q^2 B} = \left(\frac{2\pi m}{q^2}\right)\left(\frac{\frac{1}{2}mv_\perp^2}{B}\right) = \frac{2\pi m}{q^2}\mu \tag{50}$$

As μ is conserved, so is the magnetic flux, Φ.

6. Toroidal magnetic confinement

One way of confining a particle is by the magnetic mirror discussed above. However, collisions will continuously scatter particles into a "loss cone" region in velocity space, and gradually the particles will be lost at both ends.

An alternative suggestion is to bend the magnetic field lines into a torus - which will remove the end losses. These ring-shaped confinement schemes are known as **tokomaks** (Russian acronym for "toroidal chamber with magnetic field coils") and **stellarators**. While there will be no end losses, the price is the need to curve the magnetic field lines.

The toroidal fields are generated by electromagnetic coils. The main problem with this setup is that the generated magnetic field is not uniform, but (for equal number of winding per unit length, n/l) is radially inhomogeneous, as can be seen by applying Ampere's law:

$$\oint H ds = 2\pi r H_t(r) = nI, \tag{51}$$

where nI is the total current, and $H = B/\mu_0$ is the magnetic field strength (μ_0 is the permeability). Hence, the toroidal magnetic flux density decreases radially,

$$B_t = \mu_0 H_t = \mu_0 \frac{nI}{2\pi r} \tag{52}$$

(see Figure 8).

In this inhomogeneous and curved magnetic field, charged particles experience first the toroidal drift (see Equation 26) (which depends on the sign of the charge) in the vertical direction. Second, the charge separation leads to the establishment of a vertical electric field, which causes a secondary $\mathbf{E}\times\mathbf{B}$ drift in the radial direction (for both electrons and ions; see Equation 14, and Figure 8, lower panel).

6.1. The tokomak and stellator principles

In order to compensate for these drifts, the solution is to twist the toroidal magnetic field lines, such that the outer field lines will become inner ones, and vice versa. This is achieved

Fig. 3.15 Simple torus with field coils that generate the toroidal magnetic field. The *inset* shows that the toroidal magnetic field strength decays as $B_t \propto 1/r$

Fig. 3.16 The toroidal drift leads to charge separation and a subsequent particle loss by E × B drift

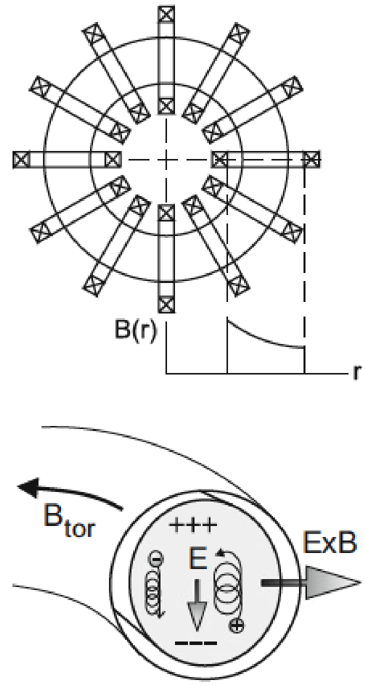

Fig. 8.— Upper: simple torus with field coils produces a toroidal magnetic field that decays as $B_t \propto r^{-1}$. Lower: This setup leads to a toroidal drift, which results in charge separation and particle loss by $\mathbf{E} \times \mathbf{B}$ drift. Figure taken from Ref. [2]

by superimposing a *poloidal magnetic field*, B_p. In tokomak, this is done by inducing a toroidal current, I_t into the plasma ring.

While tokomaks have been proven useful, there is a technical difficulty in generating strong toroidal currents over a period longer than a few tens of seconds. An alternative idea is to produce the rotational transform of the magnetic field lines by external currents. One possibility is by to use a pair of conductors that are wound in a helix around the torus. Such a device is called *stellator*, and is useful as it enables a steady-state operation.

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