Stochastic processes in plasma: diffusion, mobility, resistivity

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This part of the course is based on Refs. [1], [2] and [3].

1. Introduction

A plasma contains many particles (electrons and ions). The motion of these particles involve multiple collisions with each other (and with neutral particles that may be present). Description of this motion therefore require the use of statistical approach, as it can no longer be reduced to a deterministic motion of individual particles. We will thus use basic tools from statistical physics to describe various plasma phenomena.

2. Velocity distribution

2.1. Maxwell's velocity distribution

As we discussed in the introduction part, when the particles are in thermodynamic equilibrium, their velocity distribution is described by the Maxwell-Boltzmann (or Maxwellian) distribution,

$$f(v_x, v_y, v_z) = A \exp\left[-\frac{\frac{m}{2}(v_x^2 + v_y^2 + v_z^2)}{k_B T}\right]. \quad (1)$$

Here, m is the mass of the particles, T is the temperature, $k_B = 1.38 \times 10^{-23}$ J °k is Boltzmann's constant and A is a normalization factor. The function $f(v_x, v_y, v_z)dv_x dv_y dv_x$ is the number density of particles having velocity in the range $v_x..v_x+dv_x$, $v_y..v_y+dv_y$, $v_z..v_z+dv_z$. The total number density n is thus given by

$$n = \int_{-\infty}^{\infty} dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z f(v_x, v_y, v_z). \quad (2)$$

Using the number density, one can find the normalization factor,

$$A = n\left(\frac{m}{2\pi k_B T}\right)^{3/2}. \quad (3)$$

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We note that as the temperature increases, the width of the distribution increases as well.

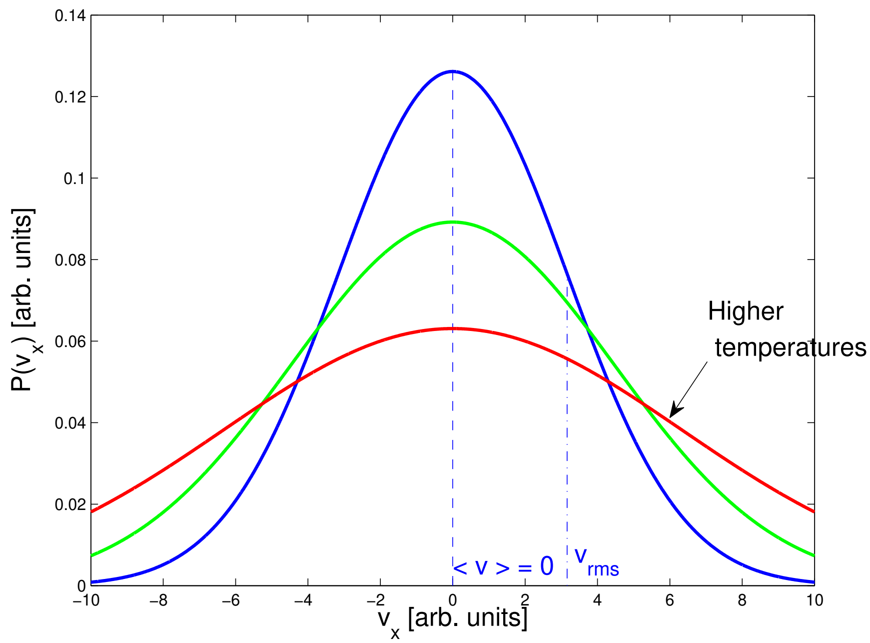

Fig. 1.— Maxwell's velocity distribution.

Sometimes we are not interested in the distribution of a specific velocity component, but rather in the distribution of the particle's **speeds** (namely, irrespective of their direction of motion). In this case, we use the **Maxwell's distribution of particles speeds**, $v = |\mathbf{v}| = (v_x^2 + v_y^2 + v_z^2)^{1/2}$. Using $d^3v = 4\pi v^2 dv$ we write Equation 1 as

$$f(v) = 4\pi v^2 n \left(\frac{m}{2\pi k_B T}\right)^{3/2} \exp\left[-\frac{mv^2}{2k_B T}\right]. \tag{4}$$

The **most probable speed** is given by setting $df(v)/dv = 0$ (namely, finding the maximum of the distribution), and is

$$v_{\max} = \sqrt{\frac{2k_B T}{m}} \tag{5}$$

(note that sometimes $v_{\max}$ is denoted as v_T)

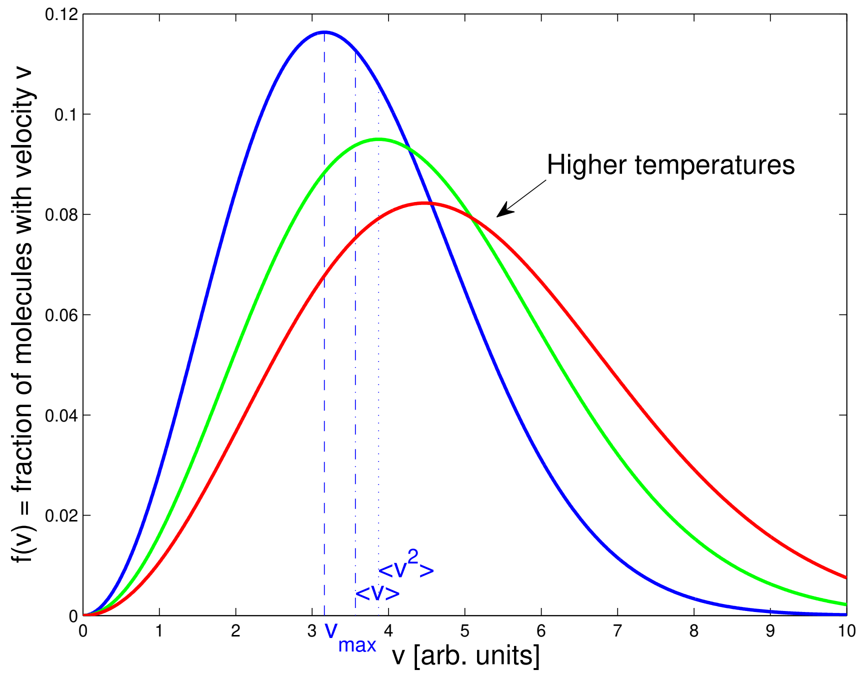

Fig. 2.— Maxwell's speed distribution.

2.2. Moments of the distribution

The **mean speed** of the gas is defined as the first moment of the distribution of speeds,

$$\langle v \rangle = \frac{1}{n}\int_0^\infty v f(v) dv = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^\infty v^3 e^{-\frac{mv^2}{2k_B T}} dv = \frac{2}{\sqrt{\pi}} v_{\rm max}. \tag{6}$$

The second moment enables to define the root-mean-square (RMS) speed, $v_{rms} = \sqrt{\langle v^2 \rangle}$,

$$\langle v^2 \rangle = \frac{1}{n}\int_0^\infty v^2 f(v) dv = 4\pi \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_0^\infty v^4 e^{-\frac{mv^2}{2k_B T}} dv \tag{7}$$

or $v_{rms} = \sqrt{\frac{3}{2}} v_{\rm max} = \sqrt{\frac{3k_B T}{m}}$.

A similar calculation gives the **mean kinetic energy** of the gas molecules,

$$\langle E_k \rangle = \frac{1}{n}\frac{m}{2}\int_0^\infty v^2 f(v) dv = \frac{3}{2} k_B T \tag{8}$$

2.3. Distribution of energies

Often it is handy to consider the distribution of particle's kinetic energies, $E_k = \frac{1}{2}mv^2$, rather than the distribution of speeds. Using $f(v)dv = f(E_k)d(E_k)$, and $d(E_k) = mvdv$, one obtains

$$f(E_k) = n\frac{2}{\sqrt{\pi}}\frac{E_k^{1/2}}{(k_BT)^{3/2}}\exp\left(-\frac{E_k}{k_BT}\right) \tag{9}$$

Note that the units of $f(E_k)$ is particles per eV interval.

3. Collisions

Collisions are unavoidable in any gas that is dense enough. During collisions particles exchange momentum and energy. Therefore, collisions affect the particles distribution. Here I will review some basics of collisions.

3.1. Cross section

The classical picture of "billiard-ball" collisions (which, somewhat surprisingly is valid under certain conditions in quantum mechanics as well) views collisions as short-range interactions between gas particles. The geometrical interpretation is that a point-like particle collides when it encounter a target which is big enough.

The concept of (effective) cross section assigns an **area** to each collision. For spherical particles of radii r_1 and r_2, the cross section is the area that results from the sum of the colliding particle's radii, $\sigma = \pi(r_1+r_2)^2$. Of course, the real situation is much more complex, and the cross section depends on various parameters, such as the particle's energy, etc.

3.2. Mean free path and collision frequency

Consider a stream of particles incidenting with velocity v upon a slab of area A and thickness dx. Inside the slab, there are N particles with density $n = N/(Adx)$ ($[m^{-3}]$). The cross section for collision is σ.

The total number of particles in the slab is

$$N = nAdx$$

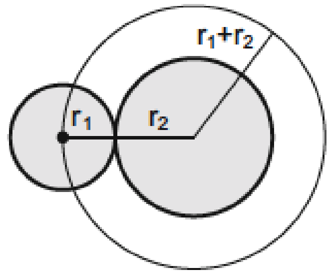

Fig. 3.— Collisions between 2 spheres (point-like particles) of radii r_1 and r_2. Figure taken from Ref. [2]

The differential probability for a collision between one particle and the particles in the slab is the ratio of the blocked area ($N\sigma$) to the total area of the slab, A,

$$dp = \frac{N\sigma}{A} = n\sigma dx. \tag{10}$$

This means that the change in the incident flux of particle as they enter the slab is

$$d\Gamma = -n\sigma\Gamma dx.$$

Note that the (number) flux Γ is the rate in which particles cross a unit area in unit time, and has units of $[\#/m^2/s]$. Integrating, we get the change in flux with distance,

$$\Gamma(x) = \Gamma_0 e^{-n\sigma x} = \Gamma_0 e^{-x/\lambda_{mfp}}. \tag{11}$$

Here, Γ_0 is the incident flux that enter the slab at $x = 0$, and

$$\boxed{\lambda_{mfp} = \frac{1}{n\sigma}} \tag{12}$$

is the **mean free path** for collisions. It represents the average distance traveled in between collisions.

The mean time between collisions is

$$\tau = \frac{\lambda_{mfp}}{v},$$

and the inverse of this is the **mean collision frequency**,

$$\nu_{coll} = \frac{v}{\lambda_{mfp}} = n\sigma v = \frac{1}{\tau}. \tag{13}$$

3.2.1. Rate coefficient

The calculation above assumed that the incoming particles are mono-energetic, with a single velocity v. In reality, the incoming particles are likely to have a Maxwellian distribution of velocities. This is important, for example, in calculating the rate at which electron beam can ionize a neutral gas.

We define the **rate coefficient** for ionization as the average of σv over the true (normalized) velocity distribution,

$$\langle \sigma v \rangle = \frac{1}{n} \int_0^\infty \sigma(v) v f(v) dv. \tag{14}$$

The rate coefficient has dimension of volume/s.

When dealing with ionization, note that the cross section for ionization is typically a function of the energy (or velocity); hence the ionization rate is determined by the overlap between the Maxwellian distribution and the ionizing cross section; see Figure 4.

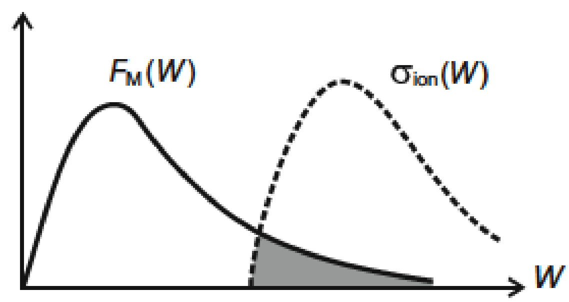

Fig. 4.— The ionization frequency is determined by the integral of the distribution function $f(W)$ (Here, W is the energy) and the ionization cross-section $\sigma(w)$. Only the shaded tail of the distribution function contributes to ionization. Figure taken from Ref. [2]

The total number of ionizing processes per unit volume per second is

$$S = n_i n_e \langle \sigma v \rangle. \tag{15}$$

3.3. Collisions in fully ionized plasmas: Coulomb collisions

Collisions between charged particles play an important role in fully ionized plasmas. Scattering of electrons by ions is responsible for the resistivity of a hot plasma. Collisions between electrons do not change the total momentum of the electron gas and therefore do not affect the conductivity. However they determine classical electron diffusion.

Here we will consider the most important type of collisions in ionized plasma - long range Coulomb collisions. Due to the Coulomb force, the approaching electron is gradually deflected by the long range Coulomb field of the ion, as is seen in Figure 5. I will not provide here the full treatment, but rather a simplified description that is sufficient for deriving the key results.

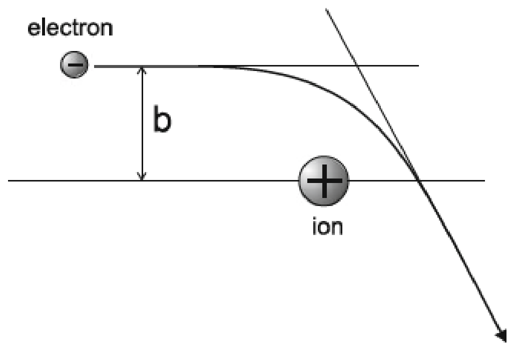

Fig. 5.— Geometry of an electron-ion collision. Figure taken from Ref. [2]

We assume that the **impact parameter**, b is smaller than the Debye length. In our simplified treatment, we may divide the electron's trajectory into 3 parts: two asymptotic motions with impact parameter b, and (nearly) circular arc. The time of interaction can be approximated by

$$\tau \approx \frac{b}{v}.$$

For large deflection angle, the change in momentum is of the same order as the electrons initial momentum, $\Delta p \approx p$. Using the formula for the Coulomb force F_c and $\Delta p \approx F_c \tau$, we have

$$\Delta p \approx p = m_e v = |F_c| \tau \approx \left(\frac{q^2}{4\pi\epsilon_0 b^2} \right) \left(\frac{b}{v} \right) = \frac{q^2}{4\pi\epsilon_0 b v} \tag{16}$$

The impact parameter for 90° scattering, $b_{\pi/2}$ is thus

$$b_{\pi/2} = \frac{q^2}{4\pi\epsilon_0 m_e v^2}. \tag{17}$$

This can be used to estimate the cross section, $\sigma_C = \pi b_{\pi/2}^2$,

$$\sigma_C \approx \frac{q^4}{16\pi\epsilon_0^2 m_e^2 v^4} \tag{18}$$

and the collision frequency,

$$\nu_{ei} = n\sigma_C v = \frac{nq^4}{16\pi\epsilon_0^2 m_e^2 v^3} \tag{19}$$

(note that $n_i = n_e = n$).

We define now the **specific resistivity**,

$$\boxed{\eta \equiv \frac{m_e \nu_{ei}}{nq^2}} \tag{20}$$

(I will show below that this definition coincides with the more familiar concept of resistivity). The specific resistivity is therefore

$$\eta = \frac{q^2}{16\pi\epsilon_0^2 m_e v^3}. \tag{21}$$

For a Maxwellian distribution of electrons, we may replace v^2 by $k_B T/m_e$, to get (to an order of magnitude),

$$\eta \approx \frac{\pi q^2 m_e^{1/2}}{(4\pi\epsilon_0)^2 (k_B T)^{3/2}}. \tag{22}$$

This estimate shows that the resistivity is independent on the electron density, and scales as $T^{-3/2}$.

The independence on the density is somewhat surprising. It can be understood as, on the one hand increasing the density increases the collision frequency, according to Equation 19, on the one hand. But at the same time it increases the number density of carriers (or the current), as in the denominator of Equation 22. The two factors exactly cancel.

In developing Equation 22 we neglected small angle collisions. When these are taken into account, we must add a logarithmic correction (first introduced by Spitzer, and known as **Spitzer resistivity**),

$$\eta \approx \frac{\pi q^2 m_e^{1/2}}{(4\pi\epsilon_0)^2 (k_B T)^{3/2}} \ln(\Lambda) \tag{23}$$

where $\ln(\Lambda) = \ln(\lambda_D/b_{\pi/2}) = \ln(9N_D)$ is called Coulomb logarithm, and N_D is the number of particles in a Debye sphere. The factor 9 comes from averaging the velocity over a Maxwellian distribution, by using Equation 7 (or Equation 8), $\langle v^2 \rangle = 3k_B T/m_e$.

A hot plasma of $T = 10$ keV has a resistivity $\eta = 5 \times 10^{-10}\ \Omega\ m$, which is lower than the resistivity of copper (at room temperature), $\eta_{Cu} = 2 \times 10^{-8}\ \Omega\ m$. This explains why a hot plasma behaves like a nearly perfect conductor

4. Transport

Under the general title of transport in plasmas, fall mobility-limited motion, conductivity and diffusion. In this section we discuss the first two, and in the next section we will discuss diffusion.

4.1. Mobility and drift velocity

Consider a gas discharge with a low degree of ionization. The motion of particles is governed by (i) the applied electric field; and (ii) collisions with the particles in the background, which cause friction (see Figure 6). Note that no external B field exists in this setup. If the degree of the ionization is low, most of the particles are atoms, and most of the collisions are elastic.

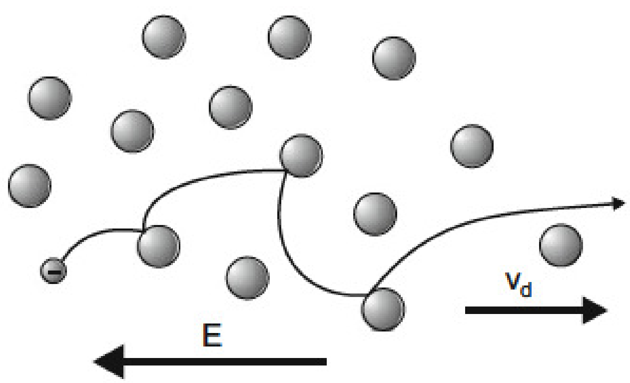

Fig. 6.— Cartoon of an electron trajectory in a homogeneous electric field. Elastic collisions with the neutral atoms interrupt the smooth trajectory. Figure taken from Ref. [2]

In a collision between a light electron and a heavy ion (or atom), the momentum transfer is small. Rather, the incoming electron experiences a random redirection of its momentum. The overall trajectory is thus a sequence of parabolic segments.

Since it is impossible to follow the trajectory of single electrons, we must average the motion of an ensemble of electrons. For hard-sphere collisions as is considered here (and discussed above), the electron completely loses its mean momentum, $m_e\langle v_e\rangle$ in each collision. Thus, averaging the equation of motion gives

$$m_e\langle \dot{v}_e\rangle = -qE - m_e\langle v_e\rangle\nu_m, \tag{24}$$

where $\nu_m = 1/\tau_{Coll}$ is the effective collision frequency (note that Equation 24 is 1-d).

The formal solution to the equation of motion is

$$\langle v_e \rangle(t) = -\frac{qE}{m\nu_m}\left[1 - e^{-\nu_m t}\right] + \langle v_e \rangle(0) e^{-\nu_m t}. \tag{25}$$

This solution has 2 parts: the first describes the approach to a terminal drift velocity,

$$v_d = -\frac{qE}{m\nu_m} = -\mu_e E. \tag{26}$$

The term $\mu_e \equiv |q|/m\nu_m$ is called **mobility**, and can be defined separately for electrons and for ions,

$$\boxed{\mu_e = \frac{q}{m_e \nu_{m,e}} \quad ; \quad \mu_i = \frac{q}{m_i \nu_{m,i}}}$$

The terminal velocity v_d is called the **drift velocity**. It is set by balance between the force exerted by the electric field, and the friction caused by the collisions.

The second term represents the loss of memory of the initial conditions (initial velocity), $\langle v_e \rangle(0)$ with time.

4.2. Electrical conductivity

The drift velocity of electrons and ions can be used to define the **electric current density**,

$$j = j_e + j_i = n\left[(-q)v_{d,e} + q v_{d,i}\right] = nq(\mu_e + \mu_i)E \equiv \sigma E. \tag{27}$$

This linear relation between the current density and the electric field is **Ohm's law**. The quantity σ is known as the **total conductivity**.

Note that the conductivity is the inverse of the specific resistivity (see Equation 20), as

$$\sigma_{e,i} = nq\mu_{e,i} = \frac{nq^2}{m_{e,i}\nu_m} \tag{28}$$

where we used $\sigma_{e,i}$ to denote the electron and ion conductivity. Thus, $\sigma_{e,i} = 1/\eta$.

5. Diffusion

In addition to the force exerted by the electric field, random motion in the presence of a density gradient will lead to a diffusion. In general, diffusion occurs when there is a gradient of concentration; it does not have to be associated with forces, as in the case, e.g., of heat flow or diffusion of neutrinos through matter. Diffusive motion will be from a region of high concentration to a region of lower concentration.

5.1. Random walk and diffusion equation in one-dimension

As a simple illustration that serves to understand the difference between deterministic motion (derived by a force) and diffusive motion, consider a random motion in a 1-d system. The particle assumes a velocity v. After each collision its velocity may change to $\pm v$, with equal probability. The time between two collisions is $\tau = \lambda_{mfp}/v$. We can thus consider a random walk with time step τ on a grid of spacing λ_{mfp}.

Denote by $P^n(x_m)$ the probability that the particles will be in location $x_m = m\lambda_{mfp}$ in time step n. Since the transition probability to each side is exactly 1/2, we can write an evolution equation for $P^n(x_m)$,

$$P^{n+1}(x_m) = \frac{1}{2}\left[P^n(x_{m-1}) + P^n(x_{m+1})\right] \tag{29}$$

We can subtract $P^n(x_m)$ on both sides,

$$P^{n+1}(x_m) - P^n(x_m) = \frac{1}{2}\left[P^n(x_{m-1}) - 2P^n(x_m) + P^n(x_{m+1})\right]. \tag{30}$$

and approximate the differences by derivatives, to get

$$\tau\frac{\partial P(x,t)}{\partial t} = \frac{\lambda_{mfp}^2}{2}\frac{\partial^2 P(x,t)}{\partial x^2}. \tag{31}$$

We can define the **diffusion coefficient**, $D \equiv \lambda_{mfp}^2/2\tau$, to write a diffusion equation for the probability

$$\frac{\partial P(x,t)}{\partial t} = D\frac{\partial^2 P(x,t)}{\partial x^2}. \tag{32}$$

A formal solution to the diffusion Equation (32) for an initial delta-function, $P(x,0) = \delta(x)$ is

$$P(x,t) = \frac{1}{(4\pi Dt)^{1/2}}\exp\left(-\frac{x^2}{4Dt}\right) \tag{33}$$

This is a Gaussian with zero mean and time-dependent width, see Figure 7.

The mean square displacement, which represents the variance σ^2 is

$$\langle x(t)^2\rangle = \int_{-\infty}^{\infty} x^2 P(x,t)dx = \sigma^2 = 2Dt \tag{34}$$

We thus find that $\sigma \propto \sqrt{t}$.

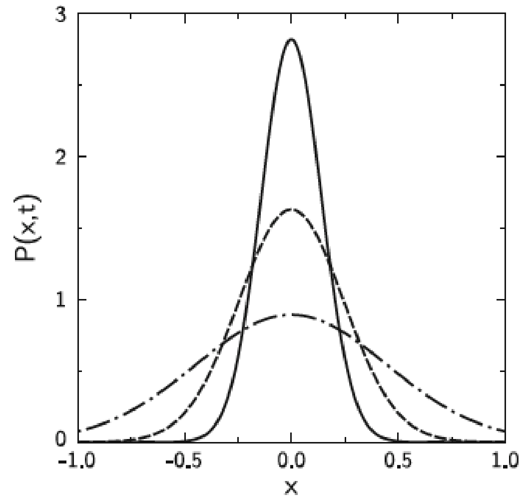

Fig. 7.— Evolution of the probability function $P(x,t)$ for $Dt = 0.01, 0.03, 0.1)$ (solid, dash, dash-dotted), respectively. Figure taken from Ref. [2]

5.2. Diffusion in a fluid: Einstein relation

The fluid equation of motion which will be discussed shortly, without magnetic field but with electric field, pressure gradient (= force) and collisions reads

$$mn\frac{d\mathbf{v}}{dt} = \pm qn\mathbf{E} - \nabla p - mn\nu\mathbf{v}. \tag{35}$$

Here, ∇p is the force due to the pressure gradient, and $-mn\nu\mathbf{v}$ is the momentum loss rate (= force) due to collisions with frequency ν.

In a steady state, we have $d\mathbf{v}/dt = 0$, and for isothermal plasma we can write $\vec{\nabla} p = k_b T \vec{\nabla} n$. We thus obtain

$$\mathbf{v} = \frac{1}{nm\nu}\left(\pm nq\mathbf{E} - k_B T\vec{\nabla} n\right) = \pm\frac{q}{m\nu}\mathbf{E} - \frac{k_B T}{m\nu}\frac{\vec{\nabla} n}{n} \tag{36}$$

We recognize the first term on the right hand side to be associated with the drift velocity (Equation 26), with the mobility $\mu = |q|/m\nu$, and the second term to be associated with a diffusion, with diffusion coefficient $\boxed{D = \frac{k_B T}{m\nu}}$.

We can therefore relate the diffusion coefficient D to the mobility μ via **Einstein rela-**

tion,

$$D = \frac{k_B T}{q}\mu \tag{37}$$

This previously unexpected relation was first derived by Einstein in his 1905 paper on Brownian motion.

Using $k_B T/m \sim v_{th}^2$, we may write

$$D = \frac{k_B T}{m\nu} \sim v_{th}^2 \tau \sim \frac{\lambda_{mfp}^2}{\tau}. \tag{38}$$

Thus form (hopefully) clarifies the random motion nature of diffusion, with a step length λ_{mfp}.

We may write the **flux density** of particles as

$$\mathbf{\Gamma} = n\mathbf{v} = \pm\mu n\mathbf{E} - D\vec{\nabla} n \tag{39}$$

Fick's first law of diffusion, which states that a flux arises from a gradient of the concentration of particles, can be viewed as a special case, where $\mathbf{E} = 0$, or neutral particles,

$$\mathbf{\Gamma} = -D\vec{\nabla} n. \tag{40}$$

We need to pay attention to a very delicate point. The gradient that appears in the diffusion equation does **not** imply that the particle interacts with other members of its own species in its random walk that produces the diffusion. Rather it implies that there is a net flux of particles from higher concentration region to lower ones, simply because more particles start in dense regions. This flux of particles is obviously proportional to the gradient of the density.

Equation 40 can in fact be put in a form similar to the diffusion Equation 32 by using the continuity equation representing particle number conservation (which we will shortly derive),

$$\frac{\partial n}{\partial t} + \vec{\nabla}(n\mathbf{v}) = 0. \tag{41}$$

Inserting the flux from equation 40, we obtain

$$\frac{\partial n}{\partial t} - D\vec{\nabla}^2 n = 0 \tag{42}$$

which is identical to Equation 32 (and is known as **Fick's second law of diffusion**).

5.3. Ambipolar diffusion

Ambipolar diffusion is diffusion of positive and negative species with opposite electrical charge due to their interaction via an electric field. The different diffusion rates of electrons and ions could lead to unequal fluxes of electrons and ions. However, if this happens, an electric field E is generated, which reduces the electron flux and increases the ion flux until they become equal: the plasma remains macroscopically neutral. The generated field is called **ambipolar electric field**.

We write the particles flux densities for this combined field-induced and diffusion processes as

$$\begin{aligned} \mathbf{\Gamma}_e &= +\mu_e n_e \mathbf{E} - D_e \vec{\nabla} n_e \\ \mathbf{\Gamma}_i &= -\mu_i n_i \mathbf{E} - D_i \vec{\nabla} n_i \end{aligned} \tag{43}$$

where D_e, D_i are the diffusion coefficients of the electrons and ions.

We may neglect the small deviation from quasi neutrality and set $n_e = n_i = n$. We can further assume that the plasma is confined to a (1-d) region, hence $n(-a) = n(a) = 0$ (see Figure 8). We may assume that the electron and ion flux densities are similar, $\mathbf{\Gamma}_e = \mathbf{\Gamma}_i = \Gamma_a$, to write

$$\begin{aligned} \Gamma_a &= -n\mu_e E - D_e \tfrac{dn}{dx} \\ \Gamma_a &= +n\mu_i E - D_i \tfrac{dn}{dx}. \end{aligned} \tag{44}$$

The ambipolar flux density is thus

$$\Gamma_a = -D_a \frac{dn}{dx} \tag{45}$$

where the ambipolar diffusion coefficient is

$$D_a = \frac{D_i \mu_e + D_e \mu_i}{\mu_e + \mu_i} \tag{46}$$

and the ambipolar electric field is

$$E(x) = \frac{D_i - D_e}{\mu_i + \mu_e} \frac{1}{n} \frac{dn}{dx} \tag{47}$$

Using $D_e \gg D_i$, $\mu_e \gg \mu_i$ and Einstein's relation (Equation 37), we can estimate

$$E_a \approx -\frac{k_B T_e}{ql} \tag{48}$$

where $l = n/(dn/dx)$ is a characteristic length scale of the density gradient.

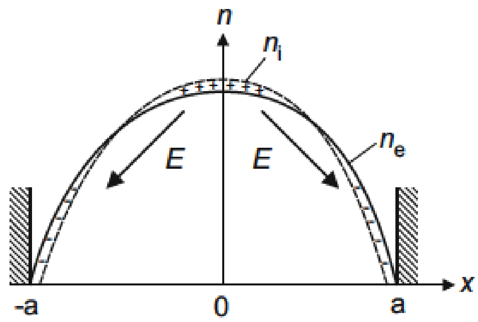

Fig. 8.— Cartoon of ion and electron density profile for ambipolar diffusion. The plasma is bounded by walls at $x = \pm a$. Figure taken from Ref. [2]

5.4. Diffusion in cylindrical geometry

A classical example is the formation of a plasma density profile in a homogeneous cylindrical tube, of length $L \gg a$, where a is the tube radius.

In a steady state, the continuity equation with a source term (which we will derive shortly) is

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\mathbf{v}) = \nu_{ion} n. \tag{49}$$

We combine this equation with Fick's law (40),

$$n\mathbf{v} = -D\vec{\nabla} n.$$

assume a steady state- hence can drop the $\partial n/\partial t$ term, and expand the ∇^2 term in cylindrical coordinates, to write

$$\frac{\partial^2 n}{\partial r^2} + \frac{1}{r}\frac{\partial n}{\partial r} + \frac{\nu_{ion}}{D} n = 0. \tag{50}$$

Here, we have explicitly assumed that there is no dependence on the z and ϕ coordinates.

The solution to Equation 50 is given by **Bessel function**, and is

$$n(r) = n(0) J_0\left(\frac{2.405 r}{a}\right), \tag{51}$$

where the factor 2.405 is the first zero of the Bessel function $J_0(x)$, and is there to ensure that the density is positive (see Figure 9).

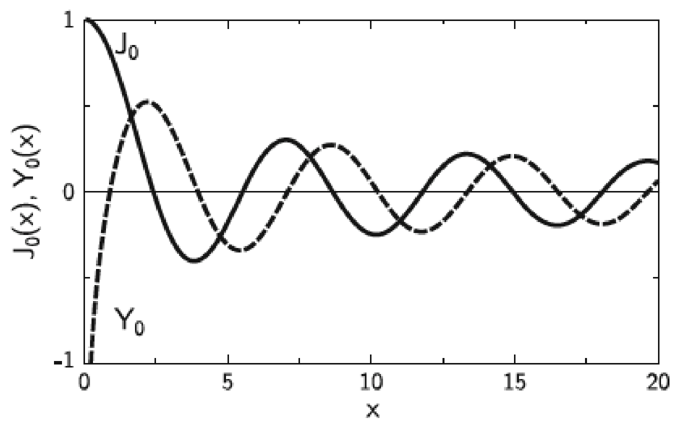

Fig. 9.— Bessel functions of first (J) and second (Y) kind of order 0. Figure taken from Ref. [2]

Inserting the solution into Equation 50, yields a relation between ν_{ion} and D,

$$\nu_{ion} = D\left(\frac{2.405}{a}\right)^2 . \tag{52}$$

This defines the steady state condition, $\nu_{ion} = \nu_{loss}$. Physically, the frequency at which plasma particles are lost by diffusion to the walls is given by the right hand side of Equation 52.

6. Motion in a magnetic field in the presence of collisions

The rate of plasma loss by diffusion can be decreased by a magnetic field. We have seen that, without collisions, an ion crossing the magnetic field performs a $\mathbf{E} \times \mathbf{B}$ drift. We now consider an ion motion in the presence of collisions. In the presence of collisions, particles migrate across $\mathbf{B}$ to the walls along the gradients, by a random walk process: in a collision, the guiding center shifts position.

Let us write the perpendicular component of the particle's equation of motion:

$$mn\frac{d\mathbf{v}_\perp}{dt} = \pm qn(\mathbf{E} + \mathbf{v}_\perp \times \mathbf{B}) - mn\nu\mathbf{v} = 0 \tag{53}$$

(in a steady state; see Equation 24). In components:

$$\begin{aligned} mn\nu v_x &= \pm qnE_x \pm qnv_yB \\ mn\nu v_y &= \pm qnE_y \mp qnv_xB \end{aligned} \tag{54}$$

(where we assumed $\mathbf{B} = B_z$).

Using the definition of the mobility $\mu = q/m\nu$ and $\omega_c = qB/m$, we write

$$\begin{aligned} v_x &= \pm\mu E_x \pm \tfrac{\omega_c}{\nu} v_y \\ v_y &= \pm\mu E_y \mp \tfrac{\omega_c}{\nu} v_x \end{aligned} \tag{55}$$

Substituting, we can solve for v_x and v_y:

$$\begin{aligned} v_x \left[1 + \left(\tfrac{\omega_c}{\nu}\right)^2\right] &= \pm\mu E_x + \left(\tfrac{\omega_c}{\nu}\right)^2 \tfrac{E_y}{B}, \\ v_y \left[1 + \left(\tfrac{\omega_c}{\nu}\right)^2\right] &= \pm\mu E_y - \left(\tfrac{\omega_c}{\nu}\right)^2 \tfrac{E_x}{B} \end{aligned} \tag{56}$$

where we used the fact that $\omega_c/\nu = \mu B$. The parameter ω_c/ν is known as the **Hall parameter**, which describes the number of gyro-periods between two collisions. When $\omega_c/\nu \gg 1$, the particle experiences only few collisions, and the velocity approaches the $\mathbf{E} \times \mathbf{B}$ drift velocity. However, when $\omega_c/\nu \ll 1$, the particles motion is in the direction of the $\mathbf{E}$ field, and approaches the collisional result $\mathbf{v} = \pm\mu\mathbf{E}$ of the unmagnetized plasma (see Equation 26).

Rather than velocities, we can consider the current densities, which lead to the **conductivity tensor** in a collisional, magnetized plasma:

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \sigma \begin{pmatrix} \frac{1}{1+(\omega_c/\nu)^2} & \frac{\omega_c/\nu}{1+(\omega_c/\nu)^2} & 0 \\ -\frac{\omega_c/\nu}{1+(\omega_c/\nu)^2} & \frac{1}{1+(\omega_c/\nu)^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{57}$$

Here, we used the fact that $\vec{j} = qn\mathbf{v}$, and $\sigma = qn\mu = \frac{q^2 n}{m\nu}$ is the conductivity in an unmagnetized plasma.

The current in the direction of the electric field is called the **Pederson current**, and the cross-field current is the **Hall current**. The current along the magnetic field is the same as in the unmagnetized case.

In an unmagnetized plasma, $\omega_c \ll \nu$, Equation 57 becomes

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \sigma \begin{pmatrix} 1 & \omega_c/\nu & 0 \\ -\omega_c/\nu & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{58}$$

while in highly magnetized plasma $\omega_c \gg \nu$, we get

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \sigma \begin{pmatrix} 0 & \nu/\omega_c & 0 \\ -\nu/\omega_c & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{59}$$

We may use $\sigma\nu/\omega_c = \frac{q^2 n}{m\nu}\frac{\nu m}{qB} = \frac{qn}{B}$, which is the same as the $\mathbf{E} \times \mathbf{B}$ drift studied earlier.

6.1. Diffusion across a magnetic field

In the presence of a density gradient, Equation 53 is modified, by adding the gradient term,

$$mn\frac{d\mathbf{v}_\perp}{dt} = \pm qn(\mathbf{E} + \mathbf{v}_\perp \times \mathbf{B}) - k_B T\vec{\nabla}n - mn\nu\mathbf{v} = 0 \tag{60}$$

Proceeding along the steps that led to Equation 56, the modified equations would read

$$\begin{aligned} v_x &= \pm\mu E_x \pm \tfrac{\omega_c}{\nu} v_y - \tfrac{D}{n}\tfrac{\partial n}{\partial x} \\ v_y &= \pm\mu E_y \mp \tfrac{\omega_c}{\nu} v_x - \tfrac{D}{n}\tfrac{\partial n}{\partial y} \end{aligned} \tag{61}$$

Substituting, we find

$$\begin{aligned} v_x\left[1 + \left(\tfrac{\omega_c}{\nu}\right)^2\right] &= \pm\mu E_x + \left(\tfrac{\omega_c}{\nu}\right)^2 \tfrac{E_y}{B} - \tfrac{D}{n}\tfrac{\partial n}{\partial x} \mp \left(\tfrac{\omega_c}{\nu}\right)^2 \tfrac{k_B T}{qB}\tfrac{1}{n}\tfrac{\partial n}{\partial y}, \\ v_y\left[1 + \left(\tfrac{\omega_c}{\nu}\right)^2\right] &= \pm\mu E_y - \left(\tfrac{\omega_c}{\nu}\right)^2 \tfrac{E_x}{B} - \tfrac{D}{n}\tfrac{\partial n}{\partial y} \pm \left(\tfrac{\omega_c}{\nu}\right)^2 \tfrac{k_B T}{qB}\tfrac{1}{n}\tfrac{\partial n}{\partial x}, \end{aligned} \tag{62}$$

The last terms on the right hand side are known as the **diamagnetic drift**. As opposed to the drifts we learned earlier, this is not a guiding center drift, but rather, the pressure gradient causes more particles to move in one direction than in the other.

For the one-before-last terms, we can define the perpendicular diffusion coefficient,

$$D_\perp = \frac{D}{1 + \left(\frac{\omega_c}{\nu}\right)^2} \tag{63}$$

We thus find the the diffusion parallel to the density gradient is reduced by a factor $1 + \left(\frac{\omega_c}{\nu}\right)^2$ relative to the $B = 0$ case.

In the limit of highly magnetized plasma, $\omega_c/\nu \gg 1$, we get

$$D_\perp \approx \frac{k_B T}{m\nu}\frac{\nu^2}{\omega_c^2} = \frac{k_B T\nu}{m\omega_c^2} \sim v_{th}^2 \frac{r_L^2}{v_{th}^2}\nu \sim r_L^2 \nu = \frac{r_L^2}{\tau}. \tag{64}$$

Comparing to Equation 38, we find that the perpendicular diffusion is a random walk process, with step length r_L rather than λ_{mfp}.

6.2. Bohm diffusion

The perpendicular diffusion coefficient, $D_\perp \propto B^{-2}$. This can be traced back to the random walk process with a step length r_L.

Although the theory was well established a long time ago, experimentally it was found (until the 1960's) that $D_\perp \propto B^{-1}$. Furthermore, the absolute value of $D_\perp$ was found to be larger than predicted. In 1949, a different, semi-empirical formula was suggested by David Bohm, Burhop and Massey:

$$D_\perp = D_B = \frac{1}{16}\frac{k_B T}{qB} \tag{65}$$

A sketch of Bohm's argument goes as follows: the underlying assumption is that a particle undergoes a single collision every gyroradius. Alternatively, that the collision frequency is equal to the gyrofrequency. In between collisions, the particle can assume to move freely, with its thermal velocity, v_{th}. The diffusion coefficient can therefore be written as (compare with Equation 38),

$$D_B \sim v_{th}^2 \tau = \frac{v_{th}^2}{\omega_c} = \frac{k_B T}{m}\frac{m}{qB} = \frac{k_B T}{qB} \tag{66}$$

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