Plasma as a fluid

Asaf Pe'er[1]

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This part of the course is based on Refs. [1] and [2].

1. Introduction

So far, we treated the motion of the plasma particles under the influence of **external** electro-magnetic field. We have already seen how reach the motion of the plasma is, under these conditions. However, the motion of charged particles themselves **create** an electro-magnetic field, which modifies the external field, and adds another layer of richness (and complexity).

Only when considering these created fields, one can construct a fully self-consistent model of plasma. Of course, we have to abandon the idea of tracking single particle trajectories (though modern computers can do it in specific cases), and instead treat the plasma as a fluid, similar to hydrodynamics.

2. The two fluid model

As explained above, we seek a description of the plasma similar to hydrodynamics, namely we study the motion of a fluid element rather than tracking the trajectories of individual particles.

2.1. Maxwell's equations

A proper description of a fluid model requires combination of Maxwell's equations with the particle current and space charges. Maxwell's equations in vacuum read

$$\vec{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{1}$$

$$\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2}$$

[1]Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

$$\vec{\nabla} \cdot \mathbf{B} = 0, \tag{3}$$

$$\vec{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \tag{4}$$

Here, $\rho = nq$ is the total charge density, and $\mathbf{j} = \sum_i qn\mathbf{v}_i$ is the current density carried by the particles.

Physically:

- Equation 1 is **Poisson's equation** that links the electric field to the space charge. It is useful for electrostatic. We may also use the electric potential Φ, which is linked to the electric field via $\mathbf{E} = -\vec{\nabla}\Phi$.
- Equation 2 is **Faraday's law of induction** (in a differential form).
- Equation 3 states the experimental fact of no magnetic monopoles (in ordinary matter).
- Equation 4 is **Ampere's law** which states that the magnetic field $\mathbf{B}$ is determined by the current $\mathbf{j}$ *and* the displacement current, $\epsilon_0 \partial \mathbf{E}/\partial t$.

Note that we did not use the magnetic field in matter, $\mathbf{H}$, or the dielectric displacement, $\mathbf{D}$. The reason is that he plasma is in fact a **diamagnet**: namely, they are repelled by an external magnetic field. This can be seen when looking at our derived result for the magnetic moment of a single particle (see "single particle motion"), where it is found to be **in the opposite direction** to the direction of the applied $\mathbf{B}$ field. Thus, a proportionality of the form $\mathbf{B} = \mu_0 \mu_r \mathbf{H}$, which is seen in ferromagnets, is not expected in plasma.

2.2. Fluid description: the continuity equation

For simplicity, let us consider first a 1-d flow, along the x direction. Consider a fixed location in space ("cell"), of size $\Delta V = \Delta x \Delta y \Delta z$. The number of particles inside the cell (i.e., in the interval $x..x + \Delta x$) is $N = nA\Delta x$, where $A = \Delta y \Delta z$ is the area of the cell perpendicular to the flow. The flux of incident particles is $I_N = nu_x A$. The flux of outgoing particles may be larger or small, and depends whether the particles velocity inside the cell changes- by acceleration or deceleration.

The change of particle number inside a cell is therefore

$$-\frac{\partial N}{\partial t} = I_N(x + \Delta x) - I_N(x) \approx \frac{\partial I_N}{\partial x} \Delta x, \tag{5}$$

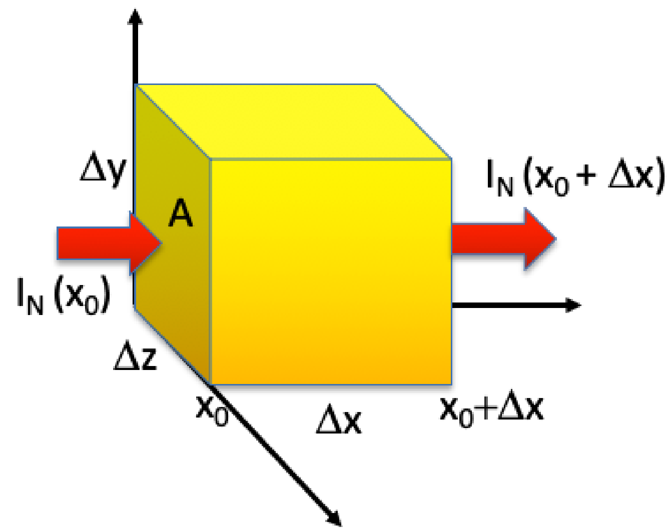

Fig. 1.— Flow inside (and outside of a cell of volume $\Delta x \Delta y \Delta z$.)

where in the last equality we Taylor expanded the particle flux. We can now divide by $\Delta V = A\Delta x$ and take the limit $\Delta V \to 0$ to write

$$\frac{\partial n}{\partial t} + \frac{\partial (n v_x)}{\partial x} = 0. \tag{6}$$

We can easily generalize this result to 3-d, which gives the **continuity equation**,

$$\boxed{\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n\mathbf{v}) = 0} \tag{7}$$

This equation describes the conservation of number of particles in the flow. If particles are generated or annihilated inside the cell (e.g., by ionization), then there is a source term - and the 0 on the right hand side is replaced with S.

A similar equation holds for the conservation of total charge. Since the charge density is $\rho = \sum_a n_a q_a$ (a represents the species), and the current density is $\mathbf{j} = \sum_a n_a q_a \mathbf{v}_a$, we get

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \mathbf{j} = 0 \tag{8}$$

This method, of fixing a cell in space and looking at the change of quantities in this cell as time passes is known as **Eulerian approach** to fluid mechanics. An alternative is the **Lagrangian approach**, where it is assumed that the observer is not fixed in space, but is fixed to an individual fluid parcel, as it moves through space and time. An easy visualization of the Lagrangian approach is to consider the point of view of someone who sits on a boat that drifts down a river.

2.3. The momentum transport (Euler) Equation

We next look at the forces that act on the particles in a given cell. The net force is the sum of all forces acting on the particles within a given cell, plus the import/export of momentum by particles entering/ leaving the cell (per unit time).

We recall Newton's equation for a single particle,

$$m\frac{d\mathbf{v}_i}{dt} = q(\mathbf{E} + \mathbf{v}_i \times \mathbf{B}). \tag{9}$$

Here, d/dt is the derivative, calculated at the position of a point-like particle, and $\mathbf{v}_i$ is the velocity vector of the particle (i).

We are now interested in the equation of motion of a fluid element. Assume first that there are no collisions or thermal motion. Then all the particles in the fluid move together, and the average velocity $\mathbf{v}$ of the particles in a fluid element is the same as that of an individual particle velocity, $\mathbf{v}_i$. The fluid equation is therefore obtained by multiplying Equation 9 with the density, n:

$$mn\frac{d\mathbf{v}}{dt} = qn(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{10}$$

However, when dealing with a fluid, we need to be careful: the time derivative is taken **at the position of the particles**, while we are interested in having an equation for a fluid element **at a fixed location in space**.

We thus need to transform the variables to a fixed observer frame, where the independent variables are (x, t) (in 1-d). Let us consider any fluid property in 1-d space, $\mathbf{G}(x, t)$. The change in time of **G in the frame moving with the fluid (=the fluid frame)** is the sum of 2 terms:

$$\frac{d\mathbf{G}(x,t)}{dt} = \frac{\partial \mathbf{G}}{\partial t} + \frac{\partial \mathbf{G}}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial \mathbf{G}}{\partial t} + v_x\frac{\partial \mathbf{G}}{\partial x} \tag{11}$$

The first term on the right represents the change in **G at a fixed point in space**, while the second term represents the change in **G** in time as the observer moves with the fluid into a region where **G** is different.

Generalizing to 3D is immediate,

$$\frac{d\mathbf{G}}{dt} = \frac{\partial \mathbf{G}}{\partial t} + (\mathbf{v} \cdot \vec{\nabla})\mathbf{G}. \tag{12}$$

This is called **convective derivative** (other names are total derivative and material derivative) in fluid dynamics, and is sometimes written $\frac{D\mathbf{G}}{Dt}$. Note that $(\mathbf{v}\cdot\vec{\nabla}) = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}$ is a **scalar** differential operator.

We thus achieved our goal - to write the derivative in the fluid frame (the Lagrangian approach), in terms of quantities known as functions of (x,t) (the Eulerian approach).

To get an insight, one can consider a man on a boat that is driven by the flow of a river from a narrow region with rapid flow speed to a wide region with slow speed. Although the flow pattern does not change in time, the man experiences a change in the boat's velocity.

In the case of a plasma, we take the quantity $\mathbf{G}$ to be the fluid velocity $\mathbf{v}$. Newton's equation (10) for fluids becomes

$$mn\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\vec{\nabla})\mathbf{v}\right] = nq\,(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{13}$$

Where $\partial\mathbf{v}/\partial t$ is the time derivative in the fixed frame.

2.3.1. The stress tensor

When thermal motions are taken into account, a pressure force has to be added to the right hand side of Equation 13. This force arises from the random motion of particles in and out of the fluid element, and therefore does not appear in the equation of motion of a single particle.

For simplicity, we will consider only the x-component of the motion through the faces $(x ... x+\Delta x)$, centered around x_0. The number of particles per second (i.e., rate of particles) crossing the face A with velocity in the range $v_x .. v_x + \Delta v_x$ is

$$\Delta I_N(v_x) = \Delta n(v_x) v_x \Delta y \Delta z, \tag{14}$$

where $\Delta n(v_x)$ is the density of particles with velocity v_x, which is given by

$$\Delta n(v_x) = \Delta v_x \int\int f(v_x, v_y, v_z)\, dv_y dv_z \tag{15}$$

Each particle carries a momentum, mv_x. The momentum flux that is carried by the crossing particles is therefore

$$\Delta I_p = (mv_x)\Delta n(v_x)|v_x|\Delta y \Delta z. \tag{16}$$

The momentum flux is the momentum transported through a boundary per unit time. The factor $|v_x|$ is a measure of the rate at which the particles pass through the boundary, and is therefore positive.

Of course, particle that move in the negative $(-x)$ direction carries away momentum from the x_0 boundary. We thus sum over all velocities, to obtain the total gain and loss for the cell located in between $x_0, x_0 + \Delta x$:

$$\begin{array}{ll} \text{Gain at } x_0: & I_p^+(x_0) = \sum_{v_x>0} \left[\Delta n(v_x)(mv_x)|v_x|\right]_{x_0} \Delta y \Delta z \\ \text{Loss at } x_0: & I_p^-(x_0) = \sum_{v_x<0} \left[\Delta n(v_x)(mv_x)|v_x|\right]_{x_0} \Delta y \Delta z \\ \text{Gain at } x_0+\Delta x_0: & I_p^-(x_0+\Delta x_0) = \sum_{v_x<0} \left[\Delta n(v_x)(mv_x)|v_x|\right]_{x_0+\Delta x_0} \Delta y \Delta z \\ \text{Loss at } x_0+\Delta x_0: & I_p^+(x_0+\Delta x_0) = \sum_{v_x>0} \left[\Delta n(v_x)(mv_x)|v_x|\right]_{x_0+\Delta x_0} \Delta y \Delta z \end{array} \tag{17}$$

We thus find that the net gain of x-momentum per unit time inside the cell is

$$\frac{\partial P_x}{\partial t} = I_p^+(x_0) - I_p^-(x_0) - I_p^+(x_0+\Delta x_0) + I_p^-(x_0+\Delta x_0) \tag{18}$$

We now Taylor expand the momentum flux and replace $|v_x| = -v_x$ when the velocity is negative, to write

$$\begin{array}{rcl} \frac{\partial P_x}{\partial t} & = & -m \sum_{v_x=-\infty}^{\infty} \left([\Delta n(v_x) v_x^2]_{x_0+\Delta x} - [\Delta n(v_x) v_x^2]_{x_0} \right) \Delta y \Delta z \\ & = & -m \frac{\partial}{\partial x} \left(n \langle v_x^2 \rangle \right) \Delta x \Delta y \Delta z \end{array} \tag{19}$$

where

$$n\langle v_x^2 \rangle = \int\int\int f(v_x, v_y, v_z) v_x^2 dv_x dv_y dv_z. \tag{20}$$

The next step is to split the particle velocities into a mean flow, u_x and a random thermal motion, $\tilde{v}_x$,

$$v_x = u_x + \tilde{v}_x. \tag{21}$$

Since the average momentum is $P_x = nmu_x \Delta V$, we write Equation 19 as

$$\frac{\partial}{\partial t}(nmu_x) = -m \frac{\partial}{\partial x} \left[n \left(u_x^2 + 2u_x \langle \tilde{v}_x \rangle + \langle \tilde{v}_x^2 \rangle \right) \right]. \tag{22}$$

For a 1-d Maxwellian distribution, we know that $(1/2)m\langle \tilde{v}_x^2 \rangle = (1/2)k_B T$. By definition, the average random motion is $\langle \tilde{v}_x \rangle = 0$. Equation 22 thus takes the form

$$\frac{\partial}{\partial t}(nmu_x) = -\frac{\partial}{\partial x} \left[nmu_x^2 + nk_B T \right]. \tag{23}$$

The first term on the right hand side of Equation 23 is the **stagnation pressure**, nmu_x^2. This is the static pressure at the stagnation point of the flow (at a stagnation point the fluid velocity is zero and all kinetic energy has been converted into pressure energy). The second term is the thermal gas pressure, $p = nk_BT$.

We can expand the left and right hand sides of Equation 23, to write

$$nm\frac{\partial u_x}{\partial t} + mu_x\frac{\partial n}{\partial t} = -mu_x\frac{\partial(nu_x)}{\partial x} - mnu_x\frac{\partial u_x}{\partial x} - \frac{\partial p}{\partial x} \tag{24}$$

Using the continuity Equation (Eq. 7), $\frac{\partial n}{\partial t} = -\frac{\partial(nu_x)}{\partial x}$, the 2nd and 3rd terms in Equation 24 cancel, and we are left with

$$mn\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x}\right) = -\frac{\partial p}{\partial x}. \tag{25}$$

This is the usual pressure-gradient force.

Adding the electromagnetic forces and generalizing to 3D, we obtain the **momentum transport equation** (neglecting collisions and viscosity),

$$\boxed{nm\left[\frac{\partial \mathbf{u}}{\partial t} + \left(\mathbf{u}\cdot\vec{\nabla}\right)\mathbf{u}\right] = nq\left(\mathbf{E} + \mathbf{u}\times\mathbf{B}\right) - \vec{\nabla}p} \tag{26}$$

2.3.2. Shear flows

So far, we have calculated the momentum exchange between neighboring cells along the mean flow. This summed up to a new net force - the pressure gradient. However, equation 26 is only a special case: the transfer of x-momentum in the x direction. Furthermore, we assumed that the flow is isotropic, so that a similar result holds in the y and z directions.

However, we can think, e.g., of transferring a y-momentum in the x direction. E.g., in a 1-d flow along the x-direction,if $u_y = 0$ at $x = x_0$ but is $\neq 0$ at $x = x_0 + \Delta x$. Then, as particles migrate across the faces A and B, they bring with them momentum in the y-direction. This is known as **shear stress**, and cannot be represented by a scalar p, but must be given by a tensor $\bar{\bar{p}}$. The components of $\bar{\bar{p}}$ are,

$$\bar{\bar{p}}_{ij} = mn\langle\tilde{v}_i\tilde{v}_j\rangle \tag{27}$$

which use the random thermal velocities that specify both the direction of motion and the direction of momentum involved.

In the general case, the gradient $\vec{\nabla}p$ is replaced by the divergence of the shear stress tensor, $-\vec{\nabla}\cdot\bar{\bar{p}}$. We note though that shear flows are associated with viscosity, which can often be neglected in plasmas.

2.4. Adiabatic process and the Equation of state

In order to complete the set of equations, we need one more relation: the equation of state. We assume an adiabatic process, and use the thermodynamic equation of state that relates the pressure p and the density,

$$p = cn^{\hat{\gamma}} \tag{28}$$

which is true for an adiabatic process. Here, C is a constant and $\hat{\gamma} = C_p/C_v$ is the ratio of specific heats. We therefore have

$$\frac{\nabla p}{p} = \hat{\gamma}\frac{\nabla n}{n} \tag{29}$$

For isothermal compression we have

$$\nabla p = \nabla(nk_BT) = k_BT\nabla n, \tag{30}$$

which is what we used so far. Thus, clearly $\hat{\gamma} = 1$ is this case.

In an adiabatic compression, the temperature also changes, and hence $\hat{\gamma} > 1$.

The validity of the equation of state requires that the heat flow be negligible, namely that the thermal conductivity is low. Fortunately, this is often the case.

2.5. The complete set of the two-fluid model equations

We consider a plasma that contains two species: electrons and ions. Each one is treated as a separate fluid. The charge ρ and current density $\mathbf{j}$ are then

$$\begin{aligned} \rho = n_iq_i + n_eq_e \quad &= (n_i - n_e)q \\ \mathbf{j} = n_iq_i\mathbf{v}_i + n_eq_e\mathbf{v}_e \quad &= (n_i\mathbf{v}_i - n_e\mathbf{v}_e)q. \end{aligned} \tag{31}$$

The momentum transfer equations for electrons and ions (neglecting collisions and viscosity) are

$$\begin{aligned} n_em_e\left[\frac{\partial\mathbf{u_e}}{\partial t} + \left(\mathbf{u_e}\cdot\vec{\nabla}\right)\mathbf{u_e}\right] &= -n_eq\left(\mathbf{E} + \mathbf{u_e}\times\mathbf{B}\right) - \vec{\nabla}p_e \\ n_im_i\left[\frac{\partial\mathbf{u_i}}{\partial t} + \left(\mathbf{u_i}\cdot\vec{\nabla}\right)\mathbf{u_i}\right] &= +n_iq\left(\mathbf{E} + \mathbf{u_i}\times\mathbf{B}\right) - \vec{\nabla}p_i \end{aligned} \tag{32}$$

Both fluids obey the equation of continuity:

$$\begin{aligned} \frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(n_e\mathbf{u_e}) = 0 \\ \frac{\partial n_i}{\partial t} + \vec{\nabla}\cdot(n_i\mathbf{u_i}) = 0. \end{aligned} \tag{33}$$

and the equations of state:

$$p_i = C_i n_i^{\hat{\gamma}_i} \quad , \quad p_e = C_e n_e^{\hat{\gamma}_e}. \tag{34}$$

Finally, we have Maxwell's equation, which we write as

$$\begin{array}{cc} \epsilon_0 \vec{\nabla} \cdot \mathbf{E} = (n_i - n_e) q, & \vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \\ \vec{\nabla} \cdot \mathbf{B} = 0, & \frac{1}{\mu_0} \vec{\nabla} \times \mathbf{B} = (n_i \mathbf{v}_i - n_e \mathbf{v}_e) q + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \end{array} \tag{35}$$

Equations 32 - 35 form a set of 16 (scalar) unknowns: $n_i, n_e, p_i, p_e, \mathbf{v}_i, \mathbf{v}_e, \mathbf{E}, \mathbf{B}$. When counting each vector equation as 3 scalar equations, we have total of 16 equations, which enable, in principle, a complete, self consistent solution.

3. The single fluid MHD equations

Let us consider a slowly evolving plasma, such that $(\mathbf{u} \cdot \vec{\nabla}) \mathbf{u}$ can be neglected. We write the momentum transport equations for electrons and ions (Equation 32) with the addition of gravitational forces and friction (collisions) between the electrons and the ions,

$$\begin{array}{l} n m_e \frac{\partial \mathbf{u_e}}{\partial t} = -nq \left(\mathbf{E} + \mathbf{u_e} \times \mathbf{B} \right) - \vec{\nabla} p_e + n m_e \mathbf{g} + n m_e \nu_{ei} (\mathbf{u}_i - \mathbf{u}_e) \\ n m_i \frac{\partial \mathbf{u_i}}{\partial t} = +nq \left(\mathbf{E} + \mathbf{u_i} \times \mathbf{B} \right) - \vec{\nabla} p_i + n m_i \mathbf{g} + n m_e \nu_{ei} (\mathbf{u}_e - \mathbf{u}_i) \end{array} \tag{36}$$

Note that conservation of momentum require that the last (friction) term be exactly opposite for ions and electrons. The momentum exchange between the electron and ion fluid is described by a collision frequency ν_{ei} and the mean exchanged momentum per volume is $n m_e \nu_{ei} (\mathbf{u}_e - \mathbf{u}_i)$. We used m_e as an approximation for the reduced mass of the colliding electron-ion pair.

Rather than direct solving the fluid equations, it is useful to transform the equations into a set of equations describing the mean mass motion, and the relative motion of the two fluids. This approach is similar to working in the center-of-mass calculation when studying relative motion.

We define the mean mass density,

$$\rho_m = n(m_i + m_e), \tag{37}$$

mean mass velocity,

$$\mathbf{v}_m = \frac{(m_i \mathbf{u}_i + m_e \mathbf{u_e})}{m_e + m_i}, \tag{38}$$

the mean current density,

$$\mathbf{j} = qn(\mathbf{u}_i - \mathbf{u}_e), \tag{39}$$

and the total pressure, $p = p_e + p_i$.

Summing up the momentum transport Equations (36) gives

$$\rho_m \frac{\partial \mathbf{v}_m}{\partial t} = \mathbf{j} \times \mathbf{B} - \vec{\nabla} p + \rho_m \mathbf{g}. \tag{40}$$

This is a single-fluid equation of motion describing the mass flow. Note that now the Lorentz force, $\mathbf{j}\times\mathbf{B}$ acts on the total current density. Furthermore, the mass motion is not affected by the friction between electron and ion fluid because it does not change the total momentum but leads only to redistribution between electron and ion fluid.

3.1. The generalized Ohm's law

We next subtract the set of Equations (36), after multiplying the top (ion) equation by m_e and the electron equation by m_i, to get

$$\begin{aligned} n m_i m_e \tfrac{\partial}{\partial t}\left(\mathbf{u}_i - \mathbf{u}_e\right) &= nq(m_e + m_i)\mathbf{E} + nq(m_e \mathbf{u}_i + m_i \mathbf{u}_e) \times \mathbf{B} \\ &\quad - m_e \vec{\nabla} p_i + m_i \vec{\nabla} p_e + n(m_e + m_i)\nu_{ei} m_e (\mathbf{u}_e - \mathbf{u}_i) \end{aligned} \tag{41}$$

We can write the mixed term as

$$\begin{aligned} m_e \mathbf{u}_i + m_i \mathbf{u}_e &= m_i \mathbf{u}_i + m_e \mathbf{u}_e - m_e(\mathbf{u}_e - \mathbf{u}_i) - m_i(\mathbf{u}_i - \mathbf{u}_e) \\ &= \tfrac{1}{n}\rho_m \mathbf{v}_m - (m_i + m_e)\tfrac{1}{nq}\mathbf{j} \end{aligned} \tag{42}$$

which is decomposition into contribution from mass motion and current density.

Equation 41 takes the form:

$$\frac{m_i m_e}{q}\frac{\partial \mathbf{j}}{\partial t} = q\rho_m\left(\mathbf{E} + \mathbf{v}_m \times \mathbf{B} - \frac{\nu_{ei} m_e}{nq^2}\mathbf{j}\right) - (m_i + m_e)\mathbf{j} \times \mathbf{B} - m_e \vec{\nabla} p_i + m_i \vec{\nabla} p_e \tag{43}$$

We divide by $m_i m_e$, and neglect terms of the order $m_e/m_i \ll 1$. In slow motion, $\partial \mathbf{j}/\partial t$ can be neglected. We thus get

$$\mathbf{E} + \mathbf{v}_m \times \mathbf{B} = \eta \mathbf{j} + \frac{1}{nq}\left(\mathbf{j} \times \mathbf{B} - \vec{\nabla} p_e\right). \tag{44}$$

Equation 44 is known as **generalized Ohm's law**. It describes the electrical properties of the conducting fluid. The term $\eta = \nu_{ei} m_e / nq^2$ is the specific resistivity we encountered earlier.

The (generalized) electric field on the left hand side, balanced the voltage drop $\eta\mathbf{j}$ due to resistivity, as well as the contribution from the Hall effect, $\mathbf{j} \times \mathbf{B}/(nq)$ and the electron pressure term, $-\vec{\nabla} p_e / nq$.

4. Magnetohydrostatics

In a static equilibria of a magnetized plasma, $\partial \mathbf{v}_m/\partial t = 0$, and the momentum transport Equation (Eq. 40) takes the form

$$0 = \mathbf{j} \times \mathbf{B} - \vec{\nabla} p + \rho_m \mathbf{g}. \tag{45}$$

Let us neglect gravitational forces which are typically small relative to magnetic forces. We can define a **magnetohydrostatic equilibrium** by

$$\mathbf{j} \times \mathbf{B} = \vec{\nabla} p. \tag{46}$$

We can dot product both sides of Equation 46 with $\mathbf{B}$. The term $\mathbf{B}\cdot(\mathbf{j}\times\mathbf{B})$ vanishes, yielding $\mathbf{B} \cdot \vec{\nabla} p = 0$: namely, $\mathbf{B}$ and $\vec{\nabla} p$ are perpendicular to each other. Similarly, $\mathbf{j} \cdot \vec{\nabla} p = 0$.

This means that both $\mathbf{B}$ and $\mathbf{j}$ must lie in a plane of constant pressure. The magnetic field lines and the current streamlines span a **magnetic surface**, which is also **isobaric**.

The same is true for $\mathbf{j}$ and $\vec{\nabla} p$. This is illustrated in Figure 2.

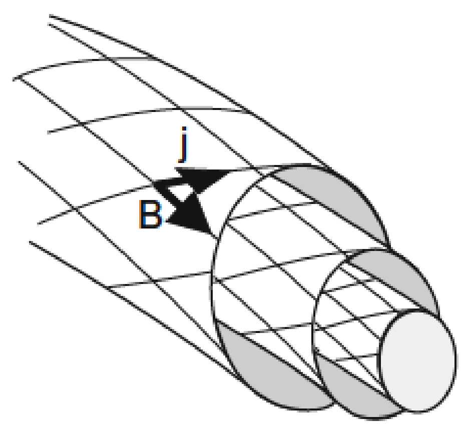

Fig. 2.— Nested magnetic surfaces in a tokomak. The force $\mathbf{j} \times \mathbf{B}$ points inward, balancing the pressure gradient. Figure taken from Ref. [2].

4.1. Magnetic pressure

We now use Ampere's law (without the displacement current),

$$\vec{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j}$$

to write

$$\mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0}(\vec{\nabla} \times \mathbf{B}) \times \mathbf{B} = -\frac{1}{\mu_0}\mathbf{B} \times (\vec{\nabla} \times \mathbf{B}) \tag{47}$$

We can use the vector identity

$$\vec{\nabla}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \vec{\nabla})\mathbf{A} + (\mathbf{A} \cdot \vec{\nabla})\mathbf{B} + \mathbf{B} \times (\vec{\nabla} \times \mathbf{A}) + \mathbf{A} \times (\vec{\nabla} \times \mathbf{B})$$

from which

$$\mathbf{B} \times (\vec{\nabla} \times \mathbf{B}) = \vec{\nabla}\left(\frac{B^2}{2}\right) - (\mathbf{B} \cdot \vec{\nabla})\mathbf{B}$$

so that

$$\vec{\nabla} p = \mathbf{j} \times \mathbf{B} = \frac{1}{\mu_0}\left[(\mathbf{B} \cdot \vec{\nabla})\mathbf{B} - \vec{\nabla}\left(\frac{B^2}{2}\right)\right] \tag{48}$$

We can now define the **magnetic pressure**,

$$\boxed{p_{mag} = \frac{B^2}{2\mu_0}} \tag{49}$$

and write Equation 48 as

$$\vec{\nabla}(p + p_{mag}) = \frac{1}{\mu_0}(\mathbf{B} \cdot \vec{\nabla})\mathbf{B} \tag{50}$$

The term on the right hand side of Equation 50 describes a force. This force arises from the mechanical tension of magnetic field lines as they curve; so this is a "curvature" force.

4.2. The pinch effect

A simple example of a plasma equilibrium is the pinch, in which plasma currents provide the magnetic pressure, which, in turn, confines (or compresses) the plasma. This field was pioneered by **Bennett**, who studied solar flares.

The pinch effect occurs when the magnetic pressure, p_{mag} exceeds the thermal pressure of the plasma, nk_BT.

4.2.1. The z-pinch

This is a very effective way of heating the plasma without carrying for long confinement times. In the z-pinch, the plasma is in contact with metallic electrodes that provides the current.

Consider a cylinder of plasma, that is bounded with radius $r = a$. There are no external fields- all the fields are produced by the plasma itself. The plasma carries a uniform current density,

$$\mathbf{j} = \begin{cases} \frac{I}{\pi a^2}\hat{z} & r < a, \\ 0 & r > a. \end{cases} \tag{51}$$

where I is the current, and a is the radius of the cylinder.

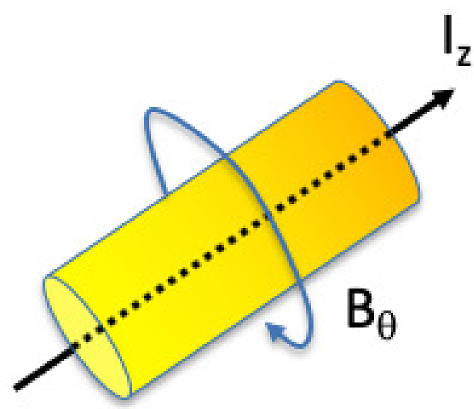

Fig. 3.— Configuration of the z-pinch.

Using Ampere's law, we have

$$\mu_0 \mathbf{j} = \vec{\nabla} \times \mathbf{B} = \left[\frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) - \frac{\partial}{\partial z}B_\theta\right]\hat{z} = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\hat{z}, \tag{52}$$

since $B_\theta = B_\theta(r)$, but is not a function of $\hat{z}$. Integration gives

$$B_\theta = \begin{cases} \mu_0 \frac{I}{2\pi a^2} r & r < a, \\ \mu_0 \frac{I}{2\pi}\frac{1}{r} & r > a. \end{cases} \tag{53}$$

In cylindrical coordinates, we have $(\mathbf{B} \cdot \vec{\nabla})\mathbf{B} = -\frac{B_\theta^2}{r}\hat{r}$. The pressure is given by the magnetostatic equilibrium Equation in the form of Equation 50,

$$\vec{\nabla}\left(p + \frac{B_\theta^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

or

$$\frac{\partial}{\partial r}\left(p+\frac{B_\theta^2}{2\mu_0}\right) = -\frac{B_\theta^2(r)}{\mu_0 r}$$

with the solution

$$p(r) = \begin{cases} \mu_0\left(\frac{I}{2\pi a^2}\right)^2(a^2-r^2) & r<a, \\ 0 & r>a. \end{cases} \tag{54}$$

At the center, $p(r=0) = nk_BT = \frac{B(a)^2}{2\mu_0}$ (there may be a factor of 2 missing somewhere).

We thus find that the temperature at the discharge axis, the radius of the compressed plasma and the total current are related via

$$T \propto \frac{I^2}{a^2}. \tag{55}$$

Equation 55 is known as **Bennett relation**. It shows how the temperature can be increased by increasing the discharge current. The temperature increases as the square of the discharge current, I.

4.2.2. The theta (θ)-pinch

This setup is, in a sense, reverse to the z-pinch setup: the magnetic field is directed along the z direction, and a large diamagnetic current directed in the θ direction.

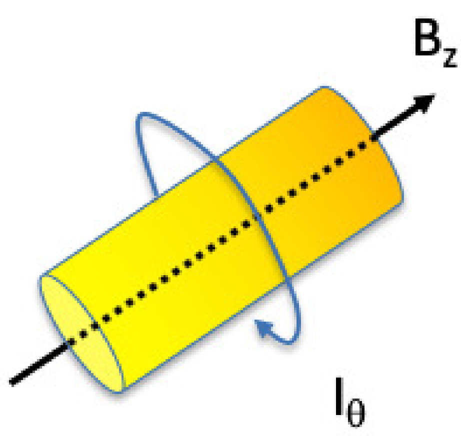

Fig. 4.— Configuration of the θ-pinch.

Since the magnetic field is directed along the z axis, $\mathbf{B} = B_z(r)$, there is no curvature, $(\mathbf{B} \cdot \vec{\nabla})\mathbf{B} = 0$. Equation 50 thus states that

$$\frac{d}{dr}\left(p + \frac{B_z^2}{2\mu_0}\right) = 0$$

This implies that the decrease of the plasma thermal pressure with radius is exactly balanced by the increase of the magnetic pressure. At radii $> a$ (a is the cylindrical radius), the thermal pressure is 0, and the magnetic field carries all the pressure.

Using the identity

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{A}(\mathbf{B} \cdot \mathbf{C}),$$

we can write the current in Equation 46 $\mathbf{j} \times \mathbf{B} = \vec{\nabla} p$ as

$$\mathbf{j} = \frac{\mathbf{B} \times \vec{\nabla} p}{B^2}$$

or

$$j_\theta = \frac{1}{B}\frac{\partial p}{\partial r}.$$

θ-pinches tend to be resistant to plasma instabilities; hopefully, we will have time to discuss these at the end of the semester.

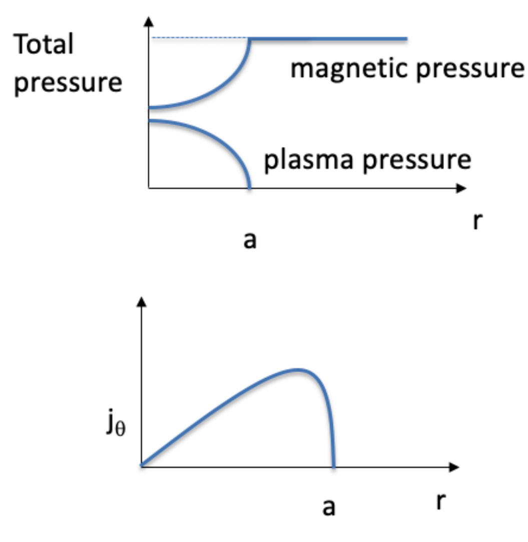

Fig. 5.— Up: pressure and down: j_θ, current in the θ-direction in the θ-pinch configuration.

Other types of pinches exist; an example is the "screw pinch". These can all be derived from the Grad-Shafranov Equation, however, it is somewhat technical, and so I am skipping it for now.

5. Magnetohydrodynamics

I the framework of the single fluid Equation of motion derived in Section 3 above, we derived a single equation that describes the center of mass motion, and another equation describing the relative motion of electrons and ions which result in the current density (Equation 39), $\mathbf{j} = nq(\mathbf{u_i} - \mathbf{u_e})$. Eventually, that led us to the generalized Ohm's law. This approach is known as magnetohydrodynamics.

5.1. Diffusion of magnetic field lines

Let us return to the generalized Ohm's law (Equation 44), and neglect the Hall term and the pressure term. We get

$$\mathbf{E} + \mathbf{v}_m \times \mathbf{B} = \eta \mathbf{j} = \frac{\eta}{\mu_0} \vec{\nabla} \times \mathbf{B} \tag{56}$$

with the use of Ampere's law.

Take the curl of Equation 56,

$$\vec{\nabla} \times \mathbf{E} + \vec{\nabla} \times (\mathbf{v}_m \times \mathbf{B}) = \frac{\eta}{\mu_0} \vec{\nabla} \times (\vec{\nabla} \times \mathbf{B})$$

And use Faraday's law of induction, $\vec{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, to write

$$-\frac{\partial \mathbf{B}}{\partial t} - \frac{\eta}{\mu_0} \vec{\nabla} \times (\vec{\nabla} \times \mathbf{B}) = -\vec{\nabla} \times (\mathbf{v}_m \times \mathbf{B}) \tag{57}$$

Consider a plasma at rest, namely $\mathbf{v}_m = 0$. Using the identity $\vec{\nabla} \times (\vec{\nabla} \times \mathbf{B}) = \vec{\nabla}(\vec{\nabla} \cdot \mathbf{B}) - \vec{\nabla}^2 \vec{B} = -\vec{\nabla}^2 \vec{B}$, Equation 57 becomes a diffusion Equation,

$$-\frac{\partial \mathbf{B}}{\partial t} + D_B \vec{\nabla}^2 \mathbf{B} = 0 \tag{58}$$

where $D_B = \eta/\mu_0$ is the diffusion coefficient. This equation thus describes a diffusion of the magnetic field lines inside the conducting medium.

We may estimate the diffusion time, by looking at the scale of change of $\mathbf{B}$, namely $\nabla^2 \mathbf{B} \to \mathbf{B}/l^2$, where l is the typical scale of change of $\mathbf{B}$. Then, $\mathbf{B}(t) \sim \exp(-t/\tau_B)$, with

$$\tau_B = \frac{\mu_0 l^2}{\eta} \tag{59}$$

Thus, as the resistivity decreases, the diffusion time gets longer. As an example, for the conditions at the metallic core of the earth, $\tau_B \approx 10^4$ years, which is the time scale in which the earth's magnetic field is observed to reverse.

We may use a similar dimensional analysis in the case where the plasma is not at rest, to estimate the relative importance of diffusion and flow:

$$\frac{\eta}{\mu_0} \vec{\nabla} \times (\vec{\nabla} \times \mathbf{B}) \approx \frac{\eta}{\mu_0} \frac{B}{l^2} \quad , \qquad \nabla \times (\mathbf{v}_m \times \mathbf{B}) \approx \frac{v_m B}{l} \tag{60}$$

which lead us to define the dimensionless **magnetic Reynolds number**,

$$R_m = \frac{\mu_0 v_m l}{\eta} \tag{61}$$

as the ratio of the mass flow to magnetic diffusion. For $R_m \gg 1$, diffusion is relatively unimportant over the length scale l, and the magnetic field lines are advected with the fluid flow. In the opposite regime, $R_m \ll 1$, diffusion is important.

5.2. Frozen-in magnetic flux

The typical conductivity of plasmas is much higher than that of metals. Thus, it is often safe to take the limit of infinite conductivity (=zero resistivity). This approximation is called **ideal MHD**. Using $\eta \to 0$ in Equation 57 gives

$$\frac{\partial \mathbf{B}}{\partial t} = \vec{\nabla} \times (\mathbf{v}_m \times \mathbf{B}) \tag{62}$$

We use the vectorial identity

$$\vec{\nabla} \times (\mathbf{v}_m \times \mathbf{B}) = (\mathbf{B} \cdot \vec{\nabla})\mathbf{v}_m - (\mathbf{v}_m \cdot \vec{\nabla})\mathbf{B} + \mathbf{v}_m(\vec{\nabla} \cdot \mathbf{B}) - \mathbf{B}(\vec{\nabla} \cdot \mathbf{v}_m)$$

(note that the 3rd term vanishes), and the continuity Equation (7), in the form

$$\vec{\nabla} \cdot \mathbf{v}_m = -\frac{1}{\rho_m} \left(\frac{\partial \rho_m}{\partial t} + (\mathbf{v}_m \cdot \vec{\nabla})\rho_m \right) = -\frac{1}{\rho_m} \frac{d\rho_m}{dt} \tag{63}$$

Where in the last equality we used the full derivative. We get

$$\frac{d\mathbf{B}}{dt} = \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}_m \cdot \vec{\nabla})\mathbf{B} = (\mathbf{B} \cdot \vec{\nabla})\mathbf{v}_m + \frac{\mathbf{B}}{\rho_m}\frac{d\rho_m}{dt} \tag{64}$$

We now use the identity

$$\frac{d}{dt}\left(\frac{\mathbf{B}}{\rho_m}\right) = \frac{1}{\rho_m}\frac{d\mathbf{B}}{dt} - \frac{\mathbf{B}}{\rho_m^2}\frac{d\rho_m}{dt} \tag{65}$$

in Equation 65 (divided by ρ_m), to get

$$\frac{d}{dt}\left(\frac{\mathbf{B}}{\rho_m}\right) = \left(\frac{\mathbf{B}}{\rho_m} \cdot \vec{\nabla}\right)\mathbf{v}_m \tag{66}$$

We thus find that when the flow velocity ($\mathbf{v}_m$) is perpendicular to the magnetic field, the right hand side vanishes, and B/ρ_m ("the number of field lines per unit mass") is conserved. This means that the motion of the mass occurs only with the magnetic field - the magnetic flux is **frozen into the plasma**.

6. Application: Alfvén waves

An excellent demonstration of the concept of frozen-in magnetic flux is low frequency waves of a magnetized plasma, known as **Alfvén waves**.

6.1. Shear Alfvénic waves

We consider ideal MHD, namely $\eta = 0$. This ensures that the internal magnetic field cannot leave the plasma by diffusion (see Equations 59, 61). We use Ampere's law,

$$\vec{\nabla} \times \mathbf{B} = \mu_0 \mathbf{j},$$

the momentum transport equation (40) (neglecting gravitational force and force due to pressure gradient),

$$\rho_m \frac{\partial \mathbf{v}_m}{\partial t} = \mathbf{j} \times \mathbf{B}$$

and the evolution of the magnetic field (Equation 62), which is written in the form

$$\frac{\partial \mathbf{B}}{\partial t} = (\mathbf{B} \cdot \vec{\nabla})\mathbf{v}_m - (\mathbf{v}_m \cdot \vec{\nabla})\mathbf{B} - \mathbf{B}(\vec{\nabla} \cdot \mathbf{v}_m). \tag{67}$$

We further assume that the flow is incompressible, namely $\vec{\nabla} \cdot \mathbf{v}_m = 0$. Using Equation 63, this implies $\rho_m =$ const.

In order to derive a wave equation, we use a **linear wave analysis**: namely, we assume that both the magnetic field and the velocity can be decomposed into a stationary equilibrium $(\mathbf{B}_0, \mathbf{v}_0)$ and a small perturbation, $\mathbf{B}_1, \mathbf{v}_1$,

$$\begin{aligned} \mathbf{B} &= \mathbf{B}_0 + \mathbf{B}_1, \\ \mathbf{v}_m &= \mathbf{v}_0 + \mathbf{v}_1. \end{aligned} \tag{68}$$

Without loss of generality, we assume that the (unperturbed) magnetic field is in the $\hat{z}$ direction, $\mathbf{B}_0 = B_0 \hat{z}$ and that the unperturbed fluid is at rest, $\mathbf{v}_0 = 0$. The momentum transport equation with the help of Ampere's law, becomes

$$\rho_m \frac{\partial \mathbf{v}_1}{\partial t} = \frac{1}{\mu_0} (\vec{\nabla} \times \mathbf{B}_1) \times \mathbf{B}_0 \tag{69}$$

(note that $\vec{\nabla} \times \mathbf{B}_0 = 0$). The evolution of the magnetic field (Equation 67) is given by

$$\frac{\partial \mathbf{B}_1}{\partial t} = (\mathbf{B}_0 \cdot \vec{\nabla}) \mathbf{v}_1 - (\mathbf{v}_1 \cdot \vec{\nabla}) \mathbf{B}_1 - \mathbf{B}_0 (\vec{\nabla} \cdot \mathbf{v}_m) = (\mathbf{B}_0 \cdot \vec{\nabla}) \mathbf{v}_1 \tag{70}$$

Where we used (i) the fact that $\mathbf{B}_0$ is fixed, so $(\mathbf{v}_1 \cdot \vec{\nabla}) \mathbf{B}_0 = (\partial \mathbf{B}_0 / \partial t) = 0$; (ii) the fact that $(\mathbf{v}_1 \cdot \vec{\nabla}) \mathbf{B}_1$ is a second order term, hence can be neglected; and (iii) the incompressibility assumption, $\vec{\nabla} \cdot \mathbf{v}_m = 0$, to remove the last term.

We seek a **perpendicular** perturbation of the magnetic field, say along the $\hat{x}$-direction, $\mathbf{B}_1 = B_1 \hat{x}$.

We use the vector identity,

$$\vec{\nabla}(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \vec{\nabla}) \mathbf{A} + (\mathbf{A} \cdot \vec{\nabla}) \mathbf{B} + \mathbf{A} \times (\vec{\nabla} \times \mathbf{B}) + \mathbf{B} \times (\vec{\nabla} \times \mathbf{A})$$

as well as $\mathbf{B} \times (\vec{\nabla} \times \mathbf{A}) = -(\vec{\nabla} \times \mathbf{A}) \times \mathbf{B}$ to write

$$(\vec{\nabla} \times \mathbf{B}_1) \times \mathbf{B}_0 = (\mathbf{B}_0 \cdot \vec{\nabla}) \mathbf{B}_1 + (\mathbf{B}_1 \cdot \vec{\nabla}) \mathbf{B}_0 + \mathbf{B}_1 \times (\vec{\nabla} \times \mathbf{B}_0) - \vec{\nabla}(\mathbf{B}_1 \cdot \mathbf{B}_0) \tag{71}$$

Since $\mathbf{B}_0$ is constant, the 2nd and 3rd terms on the right hand side vanish. Similarly, since $\mathbf{B}_1 \perp \mathbf{B}_0$, the last term vanish, and Equation 69 becomes

$$\rho_m \frac{\partial \mathbf{v}_1}{\partial t} = \frac{1}{\mu_0} (\mathbf{B}_0 \cdot \vec{\nabla}) \mathbf{B}_1 \tag{72}$$

or

$$\rho_m \frac{\partial v_{1x}}{\partial t} = \frac{B_0}{\mu_0} \frac{\partial B_{1x}}{\partial z} \tag{73}$$

Similarly, Equation 70 is written as

$$\frac{\partial B_{1x}}{\partial t} = B_0 \frac{\partial v_{1x}}{\partial z} \tag{74}$$

Equations 73, 74 can now be combined into a wave equations for the perturbed quantities,

$$\begin{aligned} \left[\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2}\right] v_{1x} = 0, \\ \left[\frac{\partial^2}{\partial t^2} - v_A^2 \frac{\partial^2}{\partial z^2}\right] B_{1x} = 0. \end{aligned} \tag{75}$$

These equations describe a transverse wave propagating along the magnetic field lines. This is a **shear Alfvén wave**, that propagates at **Alfvén speed**,

$$v_A = \left(\frac{B_0^2}{\mu_0 \rho_m}\right)^{1/2}. \tag{76}$$

Note that being transverse, the shear Alfvén wave conserves the volume between magnetic field lines (see Figure 6). Thus, there is no compression of the plasma or of the bundle of magnetic field lines.

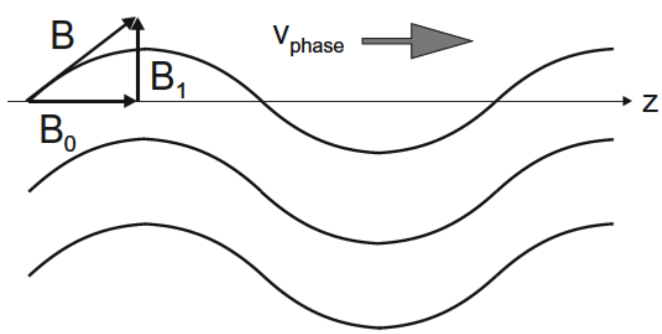

Fig. 6.— The deformed magnetic field lines of a transverse Alfvén wave propagating along the magnetic field, $\mathbf{B}_0$. Figure taken from Ref. [2]

6.2. Compressional Alfvén wave

Just for completion, I mention the fact that in addition to the shear Alfvén wave, there is also a different type of wave, that propagates across the magnetic field direction and compresses the magnetic field lines (see Figure 7). We will further discuss waves below.

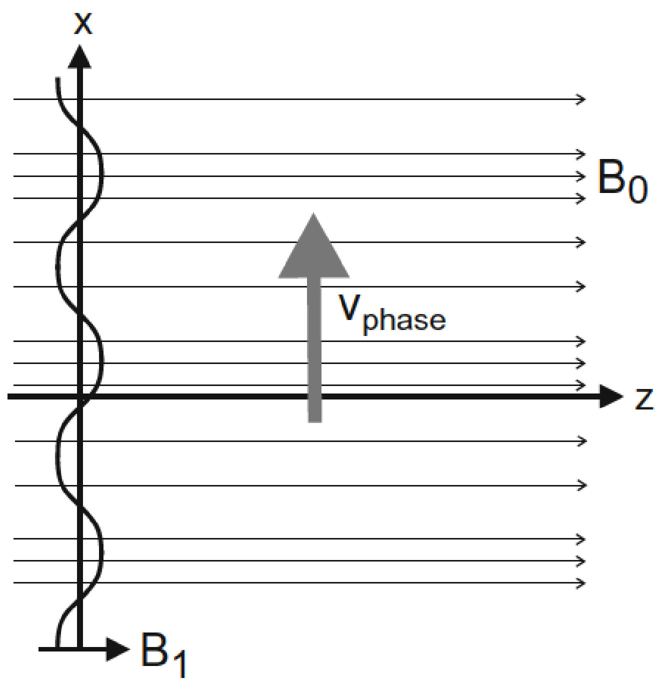

Fig. 7.— Compressional Alfvénic wave propagates across the magnetic field lines. The bouncing of the magnetic field lines can be interpreted as the superposition of a parallel perturbing field, $\mathbf{B}_1$. Figure taken from Ref. [2]

REFERENCES

[1] F. Chen, *Introduction to Plasma Physics and Controlled Fusion* (Springer), chapter 3.

[2] A. Piel, *Plasma Physics: An Introduction to Laboratory, Space, and Fusion Plasmas* (Springer), chapter 5.