Waves in plasma

Asaf Pe'er[1]

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This part of the course is based on Refs. [1] and [2].

1. Introduction

As stated in the introduction part, already Guglielmo Marconi experienced a reflection of radio waves by the earth's ionosphere. This is because the ionosphere (at altitudes 100-500 km) contains an electrically-conducting layer, that reflects radio waves - much like a mirror.

As we will shortly see, in plasma there are very many different types of waves.

2. Representation of waves; plane waves

By definition, a **wave** is an (oscillatory) perturbation of a medium, which is accompanied by transfer of energy. Since the disturbance is moving, it must be a function of both position and time, $f(\mathbf{r}, t)$. Any **periodic motion** of a fluid can be decomposed by Fourier analysis into a superposition of sinusoidal oscillations with different frequencies ω and wavelength λ. A **simple wave** contains only one of these components.

When the amplidue of the oscillation is small, the waveform is generally sinusoidal, namely it has only one component, and is called a **plane monochromatic wave**.

Any sinusoidally oscillating quantity can be written in the form

$$f = A \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)] \tag{1}$$

where (in Cartesian coordinates) $\mathbf{k} \cdot \mathbf{r} = k_x x + k_y y + k_z z$. Here, A is called the **amplitude** of the wave, and $\mathbf{k}$ is called the **propagation constant** [2].

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[1]Department of Physics, Bar-Ilan University, Ramat-Gan 52900, Israel

[2]This is likely the same meaning as the **wavevector** used in other branches of physics that deal with waves. In 1d, it is called the **wavenumber**.

The **spatial period** is known as the **wavelength**, and denoted by λ (see Figure 1). Clearly, λ has units of [length]. An increase of $|\mathbf{r}|$ by λ leaves the wave unaltered, as (in 1-d), $f(x,t) = f(x+\lambda,t)$. This implies, via Equation 1, that $|\mathbf{k}| = \frac{2\pi}{\lambda}$.

Similarly, after time $\tau = 2\pi/\omega$, one complete cycle passes through a stationary observer, and thus $f(x,t) = f(x,t+\tau)$. The time τ is known as the **temporal period** of the wave, and its inverse $\nu = 1/\tau$ is the **frequency** (which is measured in units of [Hertz], or [cycles/s]). The quantity $\omega = 2\pi\nu$ is the **angular frequency**.

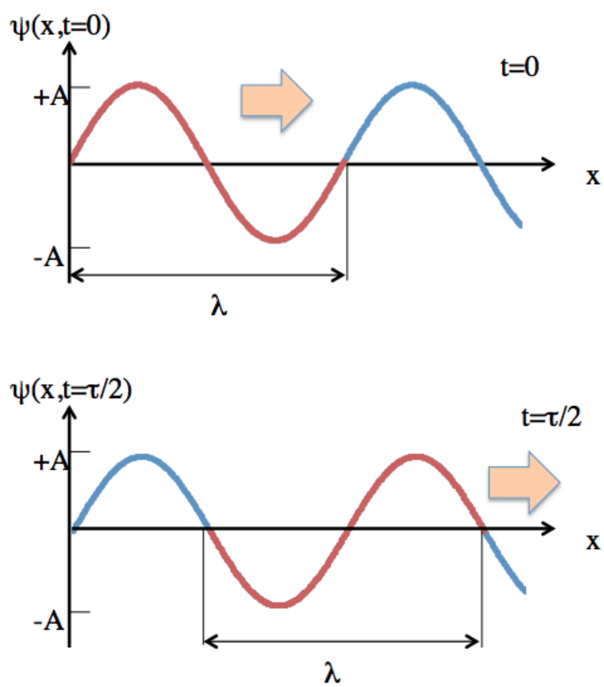

Fig. 1.— Wave

By convention, the exponential notation means that the real part of the expression is to be taken as the measurable quantity,

$$\mathrm{Re}(f) = A\cos(kx - \omega t) \tag{2}$$

2.1. Phase velocity

The argument of the cosine function, $\mathbf{k}\cdot\mathbf{r}-\omega t$ is known as the **phase** of the (monochromatic) wave. A **point of constant phase** on the wave moves such that its phase is constant, $(d/dt)(kx-\omega t)=0$, namely

$$0=\frac{d\phi}{dt}=\mathbf{k}\cdot\frac{d\mathbf{r}}{dt}-\omega. \tag{3}$$

This enables to define the **phase velocity**,

$$\mathbf{v}_\phi\equiv\left(\frac{d\mathbf{r}}{dt}\right)_\phi\equiv\frac{\omega}{k^2}\mathbf{k} \tag{4}$$

The phase velocity is a vector with magnitude $v_\phi=\omega/k$. It has the same direction as the wave vector, $\mathbf{k}$.

2.2. Wave packets and group velocity

The phase velocity of a wave in plasma can exceed the speed of light, c. This does **not** violates the theory of relativity, because an infinely long wave of constant amplitude cannot carry information.

When dealing with simple waves as is done so far, there is no need to introduce another velocity. However, often one encounters more complicated waves, such as waves that are composed by superposition of several simple waves. When superposition of simple waves occur in a localized position in space, the result is known as **wave packet**.

Wave packets are not difficult to analyze, due to the principle of superposition, from which it follows that every wave - regardless of how complicated its shape is, can be written as a superposition of simple (plane) waves. E.g., in 1d we get

$$\psi(x,t)=\int A(k)e^{i(kx-\omega(k)t)}dk. \tag{5}$$

Note that we assume an explicit dependence of the angular frequency ω on the wavenumber k, $\omega=\omega(k)$. Such a dependence is known as **dispersion relation**[3]

[3]Equally, the dispersion relation can be written as the dependence of the speed of the wave, $v=\lambda/\tau$ on the wavelength, λ, $v=v(\lambda)$. Thus, waves of different wavelength, or different frequencies, travel at different speeds.

When treating wave packets, in addition to the phase velocity of individual waves defined above, one can define the velocity of the overall shape of the wave's amplitude (also known as the *envelope* of the wave). This is known as the **group velocity**, defined by

$$\mathbf{v}_g \equiv \left(\frac{d\omega}{d\mathbf{k}}\right)_{\mathbf{k}=\mathbf{k}_0} = \left(\frac{\partial\omega}{\partial k_x}, \frac{\partial\omega}{\partial k_y}, \frac{\partial\omega}{\partial k_z}\right), \tag{6}$$

where $\mathbf{k}_0$ is the wavenumber at the center of the wavepacket.

For a simple wave, $v_g = v_\phi$. Consider, however, the simple example of two waves having slightly different wavenumbers and angular frequencies that travel together as a wavepacket:

$$\begin{aligned} \psi_1 &= A\sin\left((k+\Delta k)x - (\omega+\Delta\omega)t\right), \\ \psi_2 &= A\sin\left((k-\Delta k)x - (\omega-\Delta\omega)t\right). \end{aligned} \tag{7}$$

We have

$$\psi_1 + \psi_2 = 2A\sin(kx-\omega t)\cos\left(\Delta kx - \Delta\omega t\right), \tag{8}$$

which can be thought of as a simple wave, with varying amplitude, $2A\cos\left(\Delta kx - \Delta\omega t\right)$. In the limit $\Delta\omega, \Delta k \to 0$, one retrieves the group velocity (Equation 6), which is the velocity in which the "envelope" propagates (see Figure 2). The group velocity is the velocity in which information travels, and is always $v_g \le c$, for any physical wave packet.

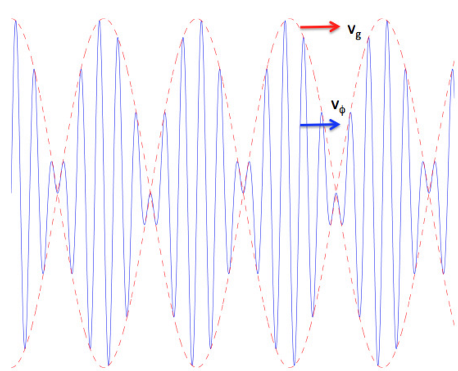

Fig. 2.— Spatial variation of the superposition of two simple waves with the same amplitude and slightly different wavenumbers reveals an "envelope" wave (red doted curve) on top of the carrier wave (blue line). The envelope travels at the group velocity.

The differences between the group velocity and the wave velocity can be understood as a property of the medium, which can be dispersive. In Figure 3, plotted is the dispersion relation, $\omega(k)$ for a dispersive medium. The phase velocity is constructed by choosing a point $(\omega, \mathbf{k})$ on the dispersion curve, and evaluating $\tan\alpha = \omega/k = v_\phi$. On the other hand, the tangent to the curve at that point have a different slope, $\tan\beta = d\omega/dk = v_g$, implyng that the phase velocity and the group velocity are different in that example. Thus, a **dispersive wave** is a wave whose phase velocity **changes** with wavelength (or frequency). In the example in Figure 3, short wavelength (with larger $\mathbf{k}$), travel at a slower speed than longer wavelengths. This is typical for most dispersive media. However, the opposite case does exist as well, in which case it is called anomalous dispersion).

In a non-dispersive medium, the dispersion relation is represented by a straight line through the origin, and $v_g = v_\phi$. All waves with different wavelengths travel at the same speed.

As a final comment, in an anisotropic medium, such as a magnetized plasma, the direction of the group velocity is not necessarily parallel to the direction of the phase velocity. We will discuss this situation below.

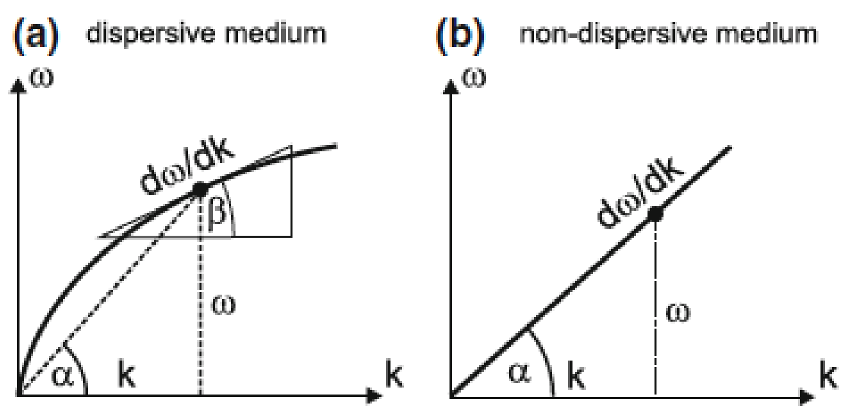

Fig. 3.— Dispersion relation in a dispersive medium (a) and non-dispersive medium (b). (a) the phase velocity is given by $\tan\alpha = \omega/k$. The slope of the tangent to the function $\omega(k)$ is the group velocity, $d\omega/dk = \tan\beta$. In a dispersive medium, $v_g \neq v_\phi$. (b) In a non-dispersive medium, $v_g = v_\phi$. Figure taken from Ref. [2]

3. Review of electromagnetic waves

As we all know and love, Maxwell's equations give rise to wave solutions. In the context of plasma physics, we will consider the interaction of plasma particles and EM waves in terms of dielectric medium: the response of plasma particles to the EM wave will be included in the dielectric constant of the plasma medium.

3.1. Basic concept

Let us begin by writing the set of Maxwell's equations,

$$\begin{aligned} \vec{\nabla} \times \mathbf{E} &= -\tfrac{\partial \mathbf{B}}{\partial t}, \\ \vec{\nabla} \times \mathbf{B} &= \mu_0 \left(\mathbf{j} + \epsilon_0 \tfrac{\partial \mathbf{E}}{\partial t} \right) . \end{aligned} \tag{9}$$

Taking the curl of the induction law, we get

$$\begin{aligned} \vec{\nabla} \times (\vec{\nabla} \times \mathbf{E}) &= -\vec{\nabla} \times \tfrac{\partial \mathbf{B}}{\partial t} \\ &= -\tfrac{\partial}{\partial t} \left(\vec{\nabla} \times \mathbf{B} \right) \\ &= -\mu_0 \epsilon_0 \tfrac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \tfrac{\partial \mathbf{j}}{\partial t}, \end{aligned} \tag{10}$$

or - using $\mu_0 \epsilon_0 = 1/c^2$,

$$\vec{\nabla} \times (\vec{\nabla} \times \mathbf{E}) + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\mu_0 \frac{\partial \mathbf{j}}{\partial t}. \tag{11}$$

The current is given by $\mathbf{j} = q(n_i \mathbf{v}_i - n_e \mathbf{v}_e)$, while the velocities can be found by solving the equations of motion for the ions and electrons, as done above. However, here, in order to discuss the propagation of waves, we need an additional relation between the **alternating** current $\mathbf{j}$ and the electric field, $\mathbf{E}$. We make the assumption that, for a given wave frequency ω, this relation is linear:

$$\mathbf{j}(\omega) = \sigma(\omega) \cdot \mathbf{E}(\omega). \tag{12}$$

Here, $\sigma(\omega)$ is a frequency-dependent conductivity. In fact $\sigma(\omega)$ contains all the microphycis of the plasma interactions.

The frequency-dependence of the physical quantities suggests that an easy way of writing the wave solutions is by using Fourier representation,

$$\begin{aligned} \mathbf{E} &= \hat{\mathbf{E}} \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)], \\ \mathbf{B} &= \hat{\mathbf{B}} \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)], \\ \mathbf{j} &= \hat{\mathbf{j}} \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]. \end{aligned} \tag{13}$$

where $\mathbf{k}$ is the wave vector.

In general, the amplitudes $\hat{\mathbf{E}}$ and $\hat{\mathbf{j}}$ are complex quantities. This gives a simple way of including a phase shift between the current density and the electric field. Both are functions of frequency and wavenumbes, $\hat{\mathbf{E}} = \hat{\mathbf{E}}(\omega, \mathbf{k})$, $\hat{\mathbf{j}} = \hat{\mathbf{j}}(\omega, \mathbf{k})$.

Using the plane wave representation in Equation 13, the partial differential operators become algebraic operations,

$$\vec{\nabla} \times \mathbf{E} \rightarrow i\mathbf{k} \times \hat{\mathbf{E}} \ , \quad \vec{\nabla} \cdot \mathbf{E} \rightarrow i\mathbf{k} \cdot \hat{\mathbf{E}} \ , \quad \frac{\partial}{\partial t} \mathbf{E} \rightarrow -i\omega \hat{\mathbf{E}}. \tag{14}$$

Maxwell's Equations (9) become

$$\begin{aligned} i\mathbf{k} \times \hat{\mathbf{E}} &= i\omega\hat{\mathbf{B}}, \\ i\mathbf{k} \times \hat{\mathbf{B}} &= \mu_0\hat{\mathbf{j}} - i\omega\mu_0\epsilon_0\hat{\mathbf{E}}. \end{aligned} \tag{15}$$

Here, the terms $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, describing the phase evolution in spae and time, cancles on both sides of the equations.

Before proceeding, I need to highlight a delicate point. When we discussed MHD equations, I argued that when writing Maxwell's equations, it is correct to use the electric field **E** and the magnetic flux density, **B**, rather than the dielectric displacement, **D** and the magnetic field strength, **H** (of course, in vacuum those are the same).

I argued that in a static situation, the dielectric displacement **D**, is not a suitable quantity to describe the plasma. The argument goes as follows: in the part about "single particle motion", we have seen that electric polarization of a plasma does only appear in time-varying fields. Any **static** polarization charges can only exist at the plasma surface, but the resulting electric field will be shielded in the plasma interior. Hence, the plasma does not behave as a ferromagnet, and should the use of **B** rater than **H** is suitable.

When discussing waves, though, the situation is different: we now make explicit use of the concept of a plasma as a dielectric medium. For example, consider the simplified picture for electron waves in a low-temperature plasma. In this picutre, ions are essentially at rest and the electrons react to the oscillating electric field. We can therefore group the plasma particles into pairs of electrons and ions that form local oscillating dipoles. These induced dipoles make the plasma a dielectric medium.

We can now give a different interpretation to the current density. When considering plasma as a dielectric medium, we can think of the wiggling motion of electrons and ions as a polarization curent, which is combined with the vacuum displacement current, $\epsilon_e(\partial\mathbf{E}/\partial t)$. In the limit of very high frequencies, only the electrons oscillate about their mean position, while the (much heavier) ions can be considered at rest. Thus, we can describe the plasma as a set of electron-ion dipoles. Such a plasma is characterized by the dielectric displacement,

$$\hat{\mathbf{D}}(\omega) = \epsilon_0\epsilon(\omega)\hat{\mathbf{E}}(\omega) \tag{16}$$

where $\epsilon(\omega)$ is known as the **dielectric constant** of the frequency ω.

For a given frequency, ω, the total displacement current is the sum of the vacuum displacement current and the conducting current ($\mathbf{j}$),

$$\frac{\partial\mathbf{D}}{\partial t} = \epsilon_0\frac{\partial\mathbf{E}}{\partial t} + \mathbf{j} \tag{17}$$

Using $\frac{\partial \mathbf{D}}{\partial t} = \epsilon_0 \epsilon(\omega) \frac{\partial \mathbf{E}}{\partial t}$, we obtain a relation between the dielectric constant $\epsilon(\omega)$ and the electric conductivity, $\sigma(\omega)$,

$$\epsilon_0 \epsilon(\omega)(-i\omega)\hat{\mathbf{E}} = \epsilon_0(-i\omega)\hat{\mathbf{E}} + \sigma(\omega)\hat{\mathbf{E}}$$

or

$$\epsilon(\omega) = 1 + \frac{i}{\omega \epsilon_0}\sigma(\omega). \tag{18}$$

In an unmagnetized plasma, $\sigma(\omega)$ and $\epsilon(\omega)$ are scalars (and functions of the frequency ω). On the other hand, when the plasma is magnetized, it becomes anisotropic, because of different motions along and across the magnetic field. In this case, the dielectric functions and conductivity become tensors,

$$\bar{\bar{\epsilon}}(\omega) = \bar{\bar{\mathbf{I}}} + \frac{\mathbf{i}}{\omega \boldsymbol{\epsilon_0}} \bar{\bar{\sigma}}(\omega) \tag{19}$$

This implies that the electric field, $\mathbf{E}$ and the curent $\mathbf{j}$ may no longer be parallel to each other.

3.2. The general dispersion relation: normal modes

In order to proceed, we use the vector identity,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

to write

$$\mathbf{k} \times (\mathbf{k} \times \hat{\mathbf{E}}) = (\mathbf{kk} - k^2\mathbf{I})\hat{\mathbf{E}} \tag{20}$$

where

$$\mathbf{kk} = \begin{pmatrix} k_x k_x & k_x k_y & k_x k_z \\ k_y k_x & k_y k_y & k_y k_z \\ k_z k_x & k_z k_y & k_z k_z \end{pmatrix}$$

is known as **dyadic product**[4].

Equation 15 becomes

$$\left[\mathbf{kk} - k^2\mathbf{I} + \frac{\omega^2}{c^2}\mathbf{I} + i\omega\mu_0\sigma(\omega)\right] \cdot \hat{\mathbf{E}} = 0 \tag{21}$$

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[4] A dyadic tensor is a second order tensor, written in a notation that fits with vector algebra. While tensor multiplication is more general than dyadics, practically, we can treat dyadics as outer product of two vectors.

or

$$\left[\mathbf{kk} - k^2\mathbf{I} + \frac{\omega^2}{c^2}\epsilon(\omega)\right] \cdot \hat{\mathbf{E}} = 0 \tag{22}$$

Equation 22 can be written explicitely, as

$$\begin{pmatrix} k_x k_x - k^2 + \frac{\omega^2}{c^2}\epsilon_{xx} & k_x k_y + \frac{\omega^2}{c^2}\epsilon_{xy} & k_x k_z + \frac{\omega^2}{c^2}\epsilon_{xz} \\ k_y k_x + \frac{\omega^2}{c^2}\epsilon_{yx} & k_y k_y - k^2 + \frac{\omega^2}{c^2}\epsilon_{yy} & k_y k_z + \frac{\omega^2}{c^2}\epsilon_{yz} \\ k_z k_x + \frac{\omega^2}{c^2}\epsilon_{zx} & k_z k_y + \frac{\omega^2}{c^2}\epsilon_{zy} & k_z k_z - k^2 + \frac{\omega^2}{c^2}\epsilon_{zz} \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0 \tag{23}$$

For $\mathbf{E} \neq 0$, the non-trivial solutions occur when the determinant of the matrix is 0. This determinant condition defines an implicit relation between frequency and wave number, which is called the **dispersion relation**,

$$D(\omega, \mathbf{k}) \equiv \mathrm{Det}\left[\mathbf{kk} - k^2\mathbf{I} + \frac{\omega^2}{c^2}\epsilon(\omega)\right] = 0. \tag{24}$$

Often one deals with a simplified case, in which $D(\omega, \mathbf{k}) = 0$ can be written in an explicit form, $\omega(\mathbf{k})$. This explicit form is also called the dispersion relation of a wave.

The expression in Equation 23 is the most general expression that describes all possible wave modes in a plasma. The specific properties of the plasma are all given by the dielectric tensor, $\epsilon(\omega)$.

In an unmagnetized plasma, the dielectric tensor reduces to a scalar function, which makes the calculation much simplified. The inclusion of a magnetic field introduces anisotropy, and requires the full tensorial notation.

4. Waves in unmagnetized plasma

Consider first the most simple case - that of waves in an unmagnetized plasma. We further consider **high frequency** waves; for these waves, ion motion can be neglected, due to the much larger ion inertia, relative to the electrons.

This can be proved as folows. The electron equation of motion is

$$m_e \frac{d\mathbf{v}_e}{dt} = -q\mathbf{E} = -q\hat{\mathbf{E}}e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}. \tag{25}$$

Note that when using the Fourier transform of $\mathbf{E}$, we have explicitely used the linear wave analysis assumption, by removing the DC (steady) part, $\mathbf{E}_0$, $\mathbf{v}_0$.

We can integrate to write $\hat{\mathbf{v}}_e = -i\frac{q}{m_e\omega}\hat{\mathbf{E}}$. Thus, the alternating current at the angular frequency ω is (using Equation 13)

$$\hat{\mathbf{j}}_e = -n_{e,0}q\hat{\mathbf{v}}_e = i\frac{n_{e,0}q^2}{m_e\omega}\hat{\mathbf{E}}. \tag{26}$$

We could have carry a similar analysis for the ions, to find that the ion current is smaller than the electron current by m_e/m_i. Thus, at high frequencies, it is safe to consider the ions as motionless. We will relax this assumption when we discuss low-frequency electrostatic waves below.

4.1. Electromagnetic waves

We consider electromagnetic waves in the limit of cold plasma, namely we neglect the pressure term. We further neglect here collisions. Without loss of generality, we choose the wave vector in the x-direction, $\mathbf{k} = (k_x, 0, 0)$. From Equations 25, 26, the current is parallel to the electric field, $\mathbf{j} \parallel \mathbf{E}$.

The conductivity tensor σ has only diagonal elements, which, from Equation 26 are

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = i\frac{n_{e,0}q^2}{m_e\omega}.$$

The dielectric tensor (Equations 18, 19) also has only diagonal components,

$$\epsilon_{xx} = \epsilon_{yy} = \epsilon_{zz} = 1 + \frac{i}{\omega\epsilon_0}\sigma(\omega) = 1 - \frac{\omega_{p,e}^2}{\omega^2}, \tag{27}$$

where

$$\boxed{\omega_{p,e} = \left(\frac{n_{0,e}q^2}{\epsilon_0 m_e}\right)^{1/2}} \tag{28}$$

is the **electron plasma frequency** which we encountered in the introduction part. In the discussion about waves here, it is the natural frequency of the electron gas.

For the given geometry, we have $k_xk_x - k^2 = 0$ and $k_y = k_z = 0$. The wave equation 23 takes the form

$$\begin{pmatrix} \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{p,e}^2}{\omega^2}\right) & 0 & 0 \\ 0 & -k^2 + \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{p,e}^2}{\omega^2}\right) & 0 \\ 0 & 0 & -k^2 + \frac{\omega^2}{c^2}\left(1 - \frac{\omega_{p,e}^2}{\omega^2}\right) \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0 \tag{29}$$

We thus have two (well, 3) cases:

1. **Longitudinal waves**, namely $\hat{E}_x \neq 0$, while $\hat{E}_y = \hat{E}_Z = 0$.

2. **Transverse waves**: $\hat{E}_x = 0$, and either $\{\hat{E}_y \neq 0\ ,\ \ \hat{E}_z = 0\}$ or $\{\hat{E}_y = 0\ ,\ \ \hat{E}_z \neq 0\}$.

Let us focus first on transverse waves, with $\hat{E}_x = \hat{E}_z = 0$. Equation 29 reduces to

$$\left(-k^2 + \frac{\omega^2 - \omega_{p,e}^2}{c^2}\right)\hat{E}_y = 0 \tag{30}$$

Since $\hat{E}_y \neq 0$, the term in parenthesis must vanish, namely,

$$\omega^2 = \omega_{p,e}^2 + k^2c^2. \tag{31}$$

The same of course it true for the $\hat{E}_z \neq 0$ case, so overall, we obtain the dispersion relation for the transverse electromagnetic waves,

$$\omega = \left(\omega_{p,e}^2 + k^2c^2\right)^{1/2}. \tag{32}$$

Note the following.

- We have explicitely assumed that $\mathbf{k}$ is in the x-direction, and $\hat{\mathbf{E}}$ is in the y-direction. Thus, from the induction Equation 15 ($i\mathbf{k} \times \hat{\mathbf{E}} = i\omega\hat{\mathbf{B}}$), there is a $\mathbf{B}$ field in the $\hat{z}$ direction associated with the wave. The transverse wave is thus an electromagnetic wave mode.

- In the limit $n_e \to 0$, $\omega_{p,e} \to 0$ and the wave dispersion relation takes the vacuum limit, $\omega = ck$. The transverse mode in an unmagnetized plasma is therefore a light wave modified by the presence of a plasma, which is treated as a dielectric medium [5].

I show in Figure 4 a plot of the dispersion relation of a transverse electromagnetic wave. The wave can only propagate for $\omega > \omega_{p,e}$. The electron plasma frequency is thus called the **cutoff frequency** of the electromagnetic mode.

In the limit of high frequencies, the dispersion curve approaches the dispersion of light in vacuum. Physically, the increase of the frequency implies a decrease of the electric current, as $\hat{j} \propto \omega^{-1}$ (see Equation 26) due to the electron inertia. Thus, the electron current does not influence the wave.

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[5] A dielectric medium is a medium (typically, an insulator) that can be polarized by an applied electric field.

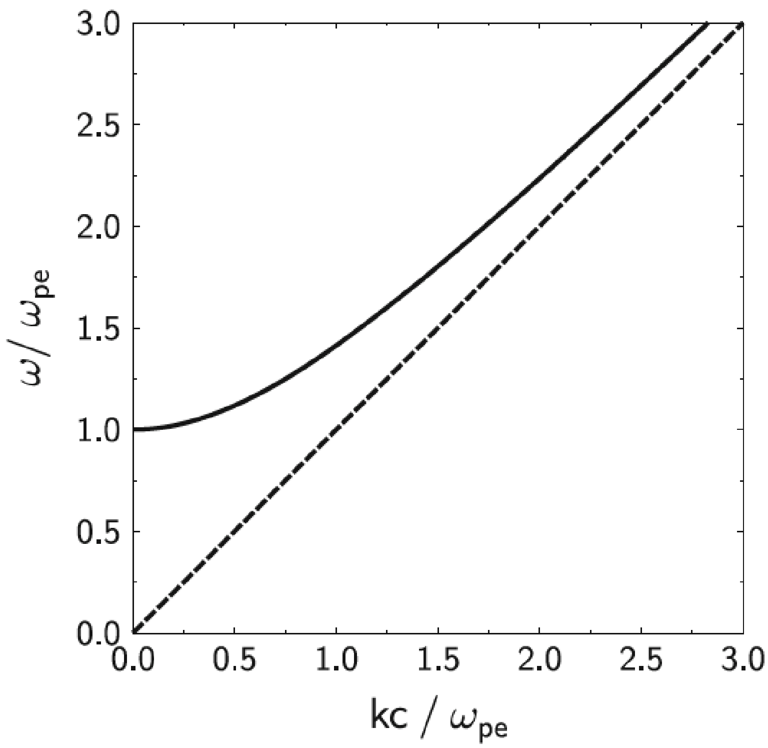

Fig. 4.— Dispersion relation for electromagnetic waves in an unmagnetized plasma. Waves propagation is only possible for frequencies larger than the plasma frequency. For $\omega \gg \omega_{p,e}$, the dispersion relation approaches that of light wave in vacuum, $\omega = kc$ (dashed line). Figure taken from Ref. [2]

The phase velocity,

$$v_\phi = \frac{\omega}{k} = \left(\frac{\omega_{p,e}^2}{k^2} + c^2\right)^{1/2} \tag{33}$$

is always $> c$. However, the group velocity,

$$v_g = \frac{d\omega}{dk} = \frac{kc^2}{\left(\omega_{p,e}^2 + k^2c^2\right)^{1/2}} \tag{34}$$

is always $< c$, so relativity is not violated.

4.1.1. The effect of collisions

So far, we have neglected collisions. This approximation can be justified as long as the wave frequency is much greater than the collision frequency. When we add collisions, the equation of motion (see, e.g., "stochastic processes" part Equation 35, and compare to Equation 25) becomes

$$m_e \frac{d\mathbf{v}_e}{dt} = -q\mathbf{E} - m_e \nu \mathbf{v},$$

where ν is the collision frequency. Fourier transforming, we get

$$m_e(-i\omega + \nu)\hat{v} = -q\hat{E} \tag{35}$$

or

$$\hat{v} = -\left[\frac{\nu}{\omega^2 + \nu^2} + \frac{i\omega}{\omega^2 + \nu^2}\right] \frac{q}{m_e} \hat{E} \tag{36}$$

namely, the electron velocity now has a real and imaginary parts with respect to the electric field, $\hat{E}$.

The real part is due to collisions, and represents a resistance; the imaginary part is a part that lags by 90° behind due to inertia (this part is similar to what we had previously, in the limit $\nu \to 0$)

Mathematically, we can follow the steps of the collision-less plasma by defining an **effective mass**, $m_e^\star$ such that $-i\omega m_e^\star = (-i\omega + \nu)m_e$, or

$$m_e^\star = m_e \left(1 + i\frac{\nu}{\omega}\right) \quad \rightarrow \quad -i\omega m_e^\star \hat{v} = -q\hat{E} \tag{37}$$

When we replace $m_e \to m_e^\star$, using Equations 31 ($k^2c^2 = \omega^2 - \omega_{p,e}^2$) with the help of the definition of the plasma frequency in Equation 28, the (complex) wavenumber becomes

$$k = \frac{1}{c}\left(\omega^2 - \frac{\omega_{p,e}^2}{1 + i(\nu/\omega)}\right)^{1/2} \tag{38}$$

I show in Figure 5 the complex dispersion relation $k(\omega)$. For weak collision frequency, $\nu \ll \omega_{p,e}$, the real part of k becomes dominant at $\omega \gtrsim \omega_{p,e}$.

For waves at freqeuncies $\omega < \omega_{p,e}$, the collisions make the plasma resistive. This explains why plasma can be generated by radio-frequency discharge: although the frequency can be much smaller than $\omega_{p,e}$, collisions imply quick dissipation of the waves, which transfer their energy that is used in heating the electron gas in the plasma.

On the other hand, at high frequencies, $\omega > 2\omega_{p,e}$, collisional damping becomes negligible.

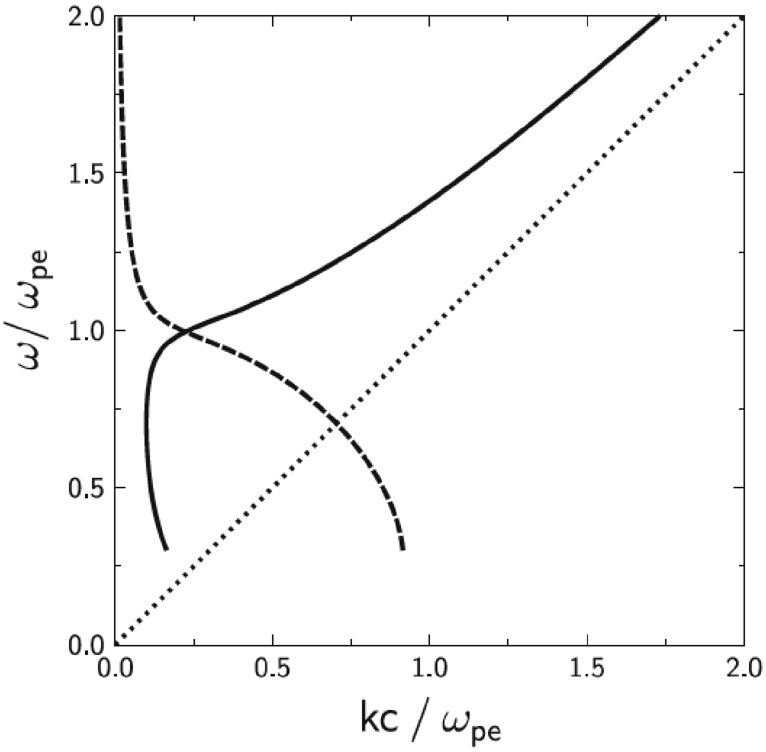

Fig. 5.— Complex dispersion relation for electromagnetic waves in an unmagnetized plasma with collision frequency, $\nu/\omega_{p,e} = 0.1$. Solid: real part of the wavenumber, dominant at $\omega > \omega_{p,e}$. Dashed: imaginary part, showing that the plasma becomes predominantly resistive at $\omega < \omega_{p,e}$. Figure taken from Ref. [2]

4.1.2. Refractive index and interferometry

In optics, the **refractive index** (or: **index of refraction**) of a transparent medium is defined as the ratio of the speed of light in vacuum c to the speed in that medium. This concept can be applied in a similar manner to electromagnetic waves in a plasma. We can define

$$\mathcal{N} = \frac{kc}{\omega} = \frac{c}{v_\phi} \tag{39}$$

For a transverse wave, we can use Equation 31, $\omega^2 = \omega_p^2 + k^2c^2$ to write

$$\mathcal{N}^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} = \epsilon(\omega).$$

Thus, by measuring the refractive index - e.g., using an interferometer, one can deduce the electron density.

For a collisionless plasma, the refractive index $\mathcal{N}$ is smaller than unity and becomes zero at the electron plasma frequency and imaginary for lower frequencies. This explains why an electromagnetic wave is reflected at the surface of a plasma when the wave frequency is lower than the electron plasma frequency.

When thinking in terms of free electrons of the silver atoms in the conduction band, this explains why a thin silver layer on a glass mirror can reflect visible light but becomes transparent in the UV range.

5. Electrostatic waves

Let us now consider the case $\mathbf{E} \parallel \mathbf{k}$ (follow Equation 29). By Faraday's law (Equation 15), $\mathbf{k} \times \mathbf{E} = 0$, and thus the magnetic field vanish. Such waves for which $\mathbf{B} = 0$ are called **electrostatic**.

From Equation 29, we have the relation

$$\frac{\omega^2}{c^2}\epsilon(\omega)\hat{E}_x = 0 \tag{40}$$

where $\epsilon(\omega) = 1 - \omega_p^2/\omega^2$. Equation 40 thus implies that $\epsilon(\omega) = 0$, which is the defining condition for the dispersion of an electrostatic wave. It further means that (in cold plasma), electrostatic waves only exist at $\omega = \omega_p$.

These are known as **Langmoir's plasma oscillations**, in which the electrons oscillate about their equilibrium at the electron plasma frequency. Note that as the dispersion relation is independent on $\mathbf{k}$, the group velocity is 0, and hence no wave packets can exist.

5.1. Electron acoustic waves

Another effect that can cause propagation of plasma oscillations is the electron thermal motion.

Since electrostatic waves are one dimensional, we may write Newton's equation of motion (Equation 25) for electrons in 1-d,

$$nm_e\frac{dv_e}{dt} = -nqE - \nabla p \tag{41}$$

From the thermodynamic equation of state, for an adiabatic and reversible process (isentropic process), we have $p = Cn^{\hat{\gamma}}$, and so $\nabla p = C\hat{\gamma}n^{\hat{\gamma}-1}\nabla n = \hat{\gamma}\frac{p}{n}\nabla n$. Adding the equation of state for an ideal gas, $p = nk_BT$, we have $\nabla p = \hat{\gamma}k_BT\nabla n$.

The adiabatic index $\hat{\gamma}$ depends on the number of degrees of freedom, N, as

$$\hat{\gamma} = \frac{2+N}{N}.$$

Thus, in 1-d, $\hat{\gamma} = 3$. Overall, we get

$$m_e\frac{dv_e}{dt} = -qE - 3\frac{k_BT}{n}\frac{\partial n}{\partial x} \tag{42}$$

We will further use the continuity equation,

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv_e) = 0. \tag{43}$$

We split the density and velocity into an equilibrium part, and a fluctuating part, in the form

$$n = n_0 + \hat{n}e^{i(kx-\omega t)} \quad , \quad v_e = v_0 + \hat{v}e^{i(kx-\omega t)}. \tag{44}$$

Without loss of generality, we can take $v_0 = 0$. The wave amplitudes, $\hat{n}$, $\hat{v}$ as well as $\hat{E}$ are first order quantities. Moving to Fourier space (see Equations 13, 14), the equation of motion (44) takes the form

$$-i\omega m_e\hat{v} = -q\hat{E} - ik\frac{3k_BT}{n_0}\hat{n} \tag{45}$$

while the continuity equation becomes

$$-i\omega\hat{n} + ikn_0\hat{v} = 0 \tag{46}$$

(recall that the term $ikv_0\hat{n} = 0$ since $v_0 = 0$, and other terms are second order). Thus,

$$\hat{v} = \frac{\omega}{k}\frac{\hat{n}}{n_0}. \tag{47}$$

For the electric field, $\hat{E}$, we use Poisson's equation, $\epsilon_0\vec{\nabla}\cdot\mathbf{E} = q(n_i - n_e)$, and note that to 0th order, $n_{i,0} = n_{e,0}$; assuming the ion density is fixed, only first order term in the electron density contributes, and we get

$$\epsilon_0 ik\hat{E} = -q\hat{n}$$

Using this result as well as Equation 47 in Equation 45, we get

$$-i\omega m_e\hat{v} = \left[\frac{q^2}{ik\epsilon_0} - ik\frac{3k_BT}{n_0}\right]\frac{kn_0}{\omega}\hat{v},$$

or

$$\omega^2 = \frac{q^2 n_0}{\epsilon_0 m_e} + k^2 \frac{3k_B T}{m_e} = \omega_{p,e}^2 + \frac{3}{2} k^2 v_{th}^2 \tag{48}$$

where we used $(1/2) m_e v_{th}^2 = k_B T$.

Equation 48 is the dispersion relation for electron aoustic waves in a warm plasma. The group velocity is given by

$$v_g = \frac{d\omega}{dk} = \left(\frac{1}{2}\right) \frac{2k \times (3/2) v_{th}^2}{\omega} = \frac{3}{2} \frac{k}{\omega} v_{th}^2 = \frac{3}{2} \frac{v_{th}^2}{v_\phi} \tag{49}$$

The dispersion relation $\omega(k)$ of the electron plasma waves is plotted in Figure 6. The slope of the curve at any point P gives the group velocity, which is clearly less then the asymptotic value of $\sqrt{3/2} v_{th}$. For non-relativistic plasma, this is always much smaller than c.

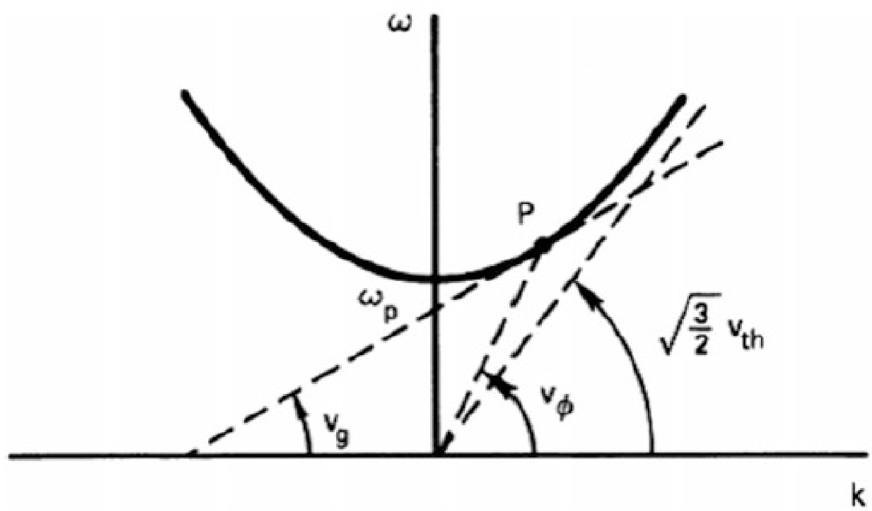

Fig. 6.— Dispersion relation for electron plasma waves. Figure taken from Ref. [1]

5.2. Ion acoustic waves

So far, we have assumed that the ions are steady, and not taking part in the wave motion. Let us now relax this assumption.

5.2.1. Sound waves

Let us first briefly review the theory of sound waves in ordinary (neutral) matter. Neglecting viscosity, the equation of motion (Navier-Stokes equation) is

$$nm\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\vec{\nabla})\mathbf{v}\right] = -\vec{\nabla}p = -\hat{\gamma}\frac{p}{n}\vec{\nabla}n \tag{50}$$

and the continuity equation (see Equation 43)

$$\frac{\partial n}{\partial t} + \vec{\nabla}(n\mathbf{v}) = 0.$$

Linearizing about a static equilibrium with uniform p_0 and n_0, we have

$$-i\omega n_0 m\hat{\mathbf{v}} = -i\mathbf{k}\hat{\gamma}\frac{p_0}{n_0}\hat{n} \tag{51}$$

and

$$-i\omega\hat{n} + in_0\mathbf{k}\cdot\hat{\mathbf{v}} = 0 \tag{52}$$

Assuming that the waves propagate in the $\hat{x}$ direciton, $\mathbf{k} = k\hat{x}$ and $\mathbf{v} = v\hat{x}$, we get

$$-i\omega n_0 m\hat{v} = -ik\hat{\gamma}\frac{p_0}{n_0}\frac{n_0 k\hat{v}}{\omega} \tag{53}$$

or

$$\frac{\omega}{k} = \left(\frac{\hat{\gamma}p_0}{mn_0}\right)^{1/2} \equiv c_s \tag{54}$$

where c_s is the speed of sound.

The sound waves are pressure waves, that propagate by collision of the (neutral) molecules.

5.2.2. Ion waves

In the absence of collisions, ordinary sound waves will not occur[6]. Ions can still transmit vibrations to each other because of their charge - the acoustic waves occur through the intermediary of an electric field. Since motion of massive ions is involved, these waves will be **low frequency** oscillations.

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[6]While we did not include the energy loss by collision term directly in Equation 50, the pressure gradient term implicitly assumed the existence of a random-walk motion of the particles.

When considering low frequency modes, we may approximate the electron mass $m_e \to 0$. This assumption is justified as, being very light (relative to the ions), the electrons mobility is huge; however, they cannot leave the region where the ions are, as this will result in strong electrostatic force, which is then balanced by the pressure gradient force. Furthermore, the huge mobility of the electrons implies that their heat conductivity is almost infinite, and we may therefore assume they are isothermal, and take $\hat{\gamma}_e = 1$.[7]

The ion and electron equations of motion are thus

$$\begin{array}{l} n_i m_i \left[\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \vec{\nabla})\mathbf{v}_i\right] = n_i q \mathbf{E} - \vec{\nabla} p_i, \\ 0 = -n_e q \mathbf{E} - \vec{\nabla} p_e \end{array} \tag{55}$$

Linearizing (using Fourier transformation), we get

$$\begin{array}{l} -i\omega m_i \hat{v}_i = q\hat{E} - \frac{ik}{n_{i,0}}(\hat{\gamma} k_B T_i)\hat{n}_i \\ 0 = -q\hat{E} - \frac{ik}{n_{e,0}}(k_B T_e)\hat{n}_e \end{array} \tag{56}$$

To these we add the continuity Equation for ions, in the form of Equation 46, (or 47 or 52),

$$-i\omega \hat{n}_i + i n_{i,0} k \hat{v}_i = 0$$

to write

$$\begin{array}{l} \hat{n}_i = \frac{n_{i,0} k q}{-i\omega^2 m_i + i k^2 \hat{\gamma} k_B T_i} \hat{E} \\ \hat{n}_e = -\frac{n_{e,0} q}{i k k_B T_e} \hat{E} \end{array} \tag{57}$$

We next use Poisson's Equation ($\epsilon_0 \vec{\nabla} \cdot \mathbf{E} = q(n_i - n_e)$; see below Equation 47),

$$ik\hat{E} = \frac{q}{\epsilon_0}(\hat{n}_i - \hat{n}_e) \tag{58}$$

to write

$$ik\hat{E} = \left(\frac{n_{i,0} q^2}{\epsilon_0 m_i}\right) \frac{k}{-i\omega^2 + ik^2 \frac{\hat{\gamma} k_B T_i}{m_i}} \hat{E} + \left(\frac{n_{e,0} q^2}{\epsilon_0 k_B T_e}\right) \frac{1}{ik} \hat{E} \tag{59}$$

Recall that the ion plasma frequency is

$$\omega_{p,i} = \left(\frac{n_{i,0} q^2}{\epsilon_0 m_i}\right)^{1/2}$$

---

[7] In a polytropic process, the plasma transits from one thermodynamic equilibrium state to another, under constant specific heat. The polytropic equation relates the plasma density n and temperature T or pressure p via $p \propto n^{\hat{\gamma}}$ or $T \propto n^{\hat{\gamma}-1}$. An isothermal process occurs at constant temperature, hence $\hat{\gamma} = 1$; for adiabatic process, $\hat{\gamma} = 5/3$, and there is no heat exchange. $\hat{\gamma} = 0$ characterizes an isobaric (=constant pressure) process. In a process that occurs at constant volume (=isochoric), the density does not change, and $\hat{\gamma} = \infty$.

and the electron Debye length is

$$\lambda_D = \left(\frac{\epsilon_0 k_B T_e}{n_{e,0} q^2} \right)^{1/2}$$

We may thus write Equation 59 as

$$1 = \frac{\omega_{p,i}^2}{\omega^2 - k^2 \frac{\hat{\gamma} k_B T_i}{m_i}} - \frac{1}{k^2 \lambda_D^2} \tag{60}$$

from which we obtain the dispersion relation for the **ion acoustic mode**,

$$\frac{\omega^2}{k^2} = \left(\frac{\hat{\gamma} k_B T_i}{m_i} + \frac{\omega_{p,i}^2 \lambda_D^2}{1 + k^2 \lambda_D^2} \right) \tag{61}$$

The dispersion curve for the ion acoustic waves is plotted in Figure 7. It is fundamentally different than the dispersion relation for the electron plasma waves (Figure 6).

For long wavelength (short wavenumber), $k\lambda_D \ll 1$, the oscillations are basically **constant velocity waves**, while for electrons the oscillations are essentially **constant frequency**.

Ion acoustic waves exist **only** when there is a thermal motion. At short wavenumber, $k\lambda_D \ll 1$, $v_g = v_\phi$.

There is a key physical difference between electron plasma waves and ion acoustic waves. In electron plasma oscillations, the ions remain essentially fixed. As opposed to that, in acoustic waves, the electrons are pulled along with the ions, and tend to shield out the electric field arising from the bouncing of the ions. The shielding is not perfect, though, due to the electron thermal motion.

The ions form regions of compression and rarefaction, similar to ordinary sound waves. The compressed region expands into the rarefaction region, by two mechanisms: 1. The ion thermal motion (the first term in Equation 61). 2. The positively charged ions disperse by the electric field, which is only partially shielded by the electrons. This leads to an interesting phenomenon, which does not exist in neutral plasma: even if the ions are cold, $T_i = 0$, ion waves still exist. In this case, the sound speed depends on the **electron** temperature and the **ion** mass, as $\omega_{p,i}\lambda_D = \left(\frac{n_i}{n_e}\right)\left(\frac{m_e}{m_i}\right)\frac{k_B T_e}{m_e}$. Furthermore, reducing the electron density (or increasing the ratio n_i/n_e) increases the phase velocity ("dusty ion acoustic waves" in Figure 7)

For most gas-discharge plasmas, $T_e \gg T_i$. In this limit, for short wavelength, $k^2\lambda_D^2 \gg 1$, we have, from Equation 61, $\omega^2 = \omega_{p,i}^2$.

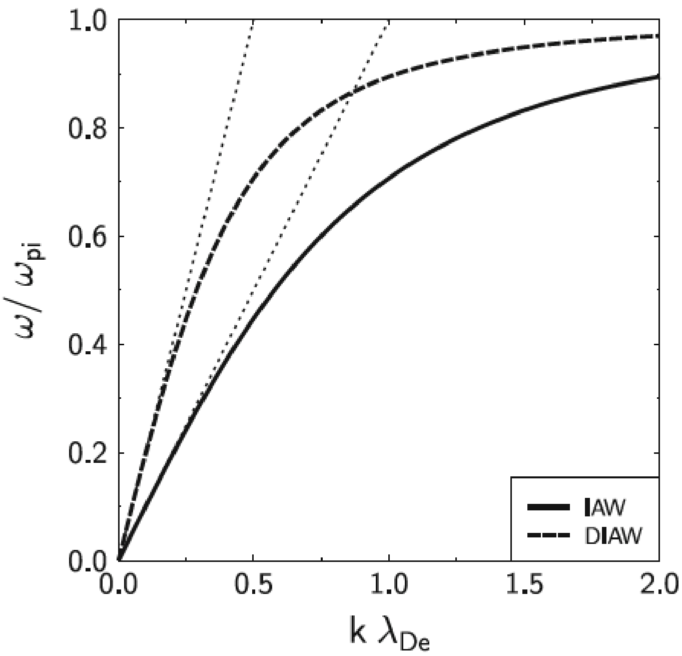

Fig. 7.— Dispersion relation for ion acoustic waves (solid) and dusty ion acoustic waves (dashed). Figure taken from Ref. [2]

6. Waves in magnetized plasma

Let us now consider the influence of an (external) magnetic field on the propagation of plasma waves. For simplicity, we restrict the discussion to cold plasmas.

We start by Newton's equation of motion for ions and electrons,

$$\frac{\partial \mathbf{v}_{i,e}}{\partial t} = \frac{\pm q}{m_{i,e}} \left(\mathbf{E}_1 + \mathbf{v}_{i,e} \times \mathbf{B}_0 \right) \tag{62}$$

where we already linearlized the equation (i.e., neglected 2nd order terms in velocity, etc.): $\mathbf{E}_1$ is the electric field associated with the wave and $\mathbf{B}_0 = (0, 0, B_0)$ is the static magnetic field.

6.1. The dielectric tensor

Using Fourier transform we have

$$\hat{v}_x = Si\frac{q}{\omega m}\left(\hat{E}_x + \hat{v}_y B_0\right) \ , \quad \hat{v}_y = Si\frac{q}{\omega m}\left(\hat{E}_y - \hat{v}_x B_0\right) \tag{63}$$

Here, I used $S = \pm 1$, where $+1$ is for ions, and -1 is for electrons; similarly, $m = m_{i,e}$, pending on whether we discuss ions or electrons. The reason for the use of this notation is to avoid confusion in what follows.

The simple way to describe the gyromotion of the particle is to use rotating vectors for the velocity and electric field:

$$\hat{v}^{\pm} = \hat{v}_x \pm i\hat{v}_y \ , \quad \hat{E}^{\pm} = \hat{E}_x \pm i\hat{E}_y \tag{64}$$

Equation 63 becomes

$$\hat{v}^{\pm} = Si\frac{q}{\omega m}\left(\hat{E}^{\pm} \mp i\hat{v}^{\pm} B_0\right) \tag{65}$$

We may use the cyclotron frequencies for ions and electrons,

$$\omega_{c,i} = \frac{qB_0}{m_i} \ , \quad \omega_{c,e} = \frac{|q|B_0}{m_e}$$

to write Equation 65 as

$$\hat{v}^{\pm} = Si\frac{q}{\omega m}\hat{E}^{\pm}\frac{1}{1 \mp S\frac{\omega_c}{\omega}} \tag{66}$$

We may transform back to Cartesian coordinates, using

$$\hat{v}_x = \frac{1}{2}\left(\hat{v}^+ + \hat{v}^-\right) \ , \quad \hat{v}_y = \frac{1}{2i}\left(\hat{v}^+ - \hat{v}^-\right)$$

to write

$$\hat{v}_x = Si\frac{q}{\omega m}\frac{\hat{E}_x + Si\frac{\omega_c}{\omega}\hat{E}_y}{1 - \left(\frac{\omega_c}{\omega}\right)^2} \ , \quad \hat{v}_y = Si\frac{q}{\omega m}\frac{-Si\frac{\omega_c}{\omega}\hat{E}_x + \hat{E}_y}{1 - \left(\frac{\omega_c}{\omega}\right)^2}$$

We may write this in a matrix form:

$$\begin{pmatrix} \hat{v}_x \\ \hat{v}_y \\ \hat{v}_z \end{pmatrix} = Si\frac{q}{\omega m}\begin{pmatrix} \frac{\omega^2}{\omega^2-\omega_c^2} & Si\frac{\omega\omega_c}{\omega^2-\omega_c^2} & 0 \\ -Si\frac{\omega\omega_c}{\omega^2-\omega_c^2} & \frac{\omega^2}{\omega^2-\omega_c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} \tag{67}$$

We may proceed by using the definition of the oscillating current, $\hat{\mathbf{j}} = \sum_\alpha n_\alpha q_\alpha \hat{\mathbf{v}}_\alpha$, where the summation of the index α is over the electrons and ions. Furthermore, recall the definition of the plasma frequency, $\omega_p^2 = nq^2/\epsilon_0 m$, so that $Si\frac{q}{m\omega} \times nq = Si\omega\epsilon_0(\omega_p^2/\omega^2)$. Using Equation 12, $\hat{j}(\omega) = \sigma(\omega)\hat{E}(\omega)$ we obtain the conductivity tensor,

$$\sigma(\omega) = i\omega\epsilon_0 \begin{pmatrix} \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2-\omega_{c,\alpha}^2} & i\sum_\alpha S(\alpha)\frac{\omega_{p,\alpha}^2}{\omega^2-\omega_{c,\alpha}^2}\frac{\omega_{c,\alpha}}{\omega} & 0 \\ -i\sum_\alpha S(\alpha)\frac{\omega_{p,\alpha}^2}{\omega^2-\omega_{c,\alpha}^2}\frac{\omega_{c,\alpha}}{\omega} & \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2-\omega_{c,\alpha}^2} & 0 \\ 0 & 0 & \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2} \end{pmatrix} \quad (68)$$

We may thus write the dielectric constant, $\epsilon(\omega) = \bar{\bar{I}} + \frac{i}{\omega\epsilon_0}\bar{\bar{\sigma}}(\omega)$ in the form

$$\epsilon(\omega) = \begin{pmatrix} S & -i\mathcal{D} & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix} \quad (69)$$

where the parameters S, P and D are

$$\begin{aligned} S &= 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2-\omega_{c,\alpha}^2} \\ D &= \sum_\alpha S(\alpha)\frac{\omega_{p,\alpha}^2}{\omega^2-\omega_{c,\alpha}^2}\frac{\omega_{c,\alpha}}{\omega} \\ P &= 1 - \sum_\alpha \frac{\omega_{p,\alpha}^2}{\omega^2} \end{aligned} \quad (70)$$

(this is known as "Stix notation"). We can now use the dielectric constant inside the general form of the wave equation (23). For that, we multiply Equation 23 by $(c/\omega)^2$, and use the definition of the refractive index, $\mathcal{N} = \frac{kc}{\omega}$. Furthermore, we assume that the wave propagates in the $x - z$ plane (due to the rotational symmetry around the direction of the magnetic field, this is a general form), so $\mathbf{k} = (k\sin\psi, 0, k\cos\psi)$. The wave equation 23 takes the form

$$\begin{pmatrix} S - \mathcal{N}^2\cos^2\psi & -iD & \mathcal{N}^2\cos\psi\sin\psi \\ iD & S - \mathcal{N}^2 & 0 \\ \mathcal{N}^2\cos\psi\sin\psi & 0 & P - \mathcal{N}^2\sin^2\psi \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0 \quad (71)$$

6.2. Circularly polarized modes

Consider first propagation of waves along the magnetic field, namely $\psi = 0$ ($\mathbf{k} \parallel \mathbf{B}_0$, or $\mathbf{k} = k\hat{z}$). The wave Equation 71 becomes

$$\begin{pmatrix} S - \mathcal{N}^2 & -iD & 0 \\ iD & S - \mathcal{N}^2 & 0 \\ 0 & 0 & P \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \\ \hat{E}_z \end{pmatrix} = 0 \quad (72)$$

We distinguish between two cases:

1. $\hat{E}_x = \hat{E}_y = 0$, $\hat{E}_z \neq 0$: We get a longitudinal wave, that is described by the dispersion relation $P = 1 - \frac{\omega_{p,e}^2 + \omega_{p,i}^2}{\omega^2} = 0$. We have encountered this before - this is the plasma oscillations that appear in the non-magnetized case (section 5, Equation 40). The magnetic field has no effect on the wave, as the oscillations are aligned in the direction of the magnetic field, and the Lorentz force vanish.

2. $\hat{E}_z = 0$, and $\hat{E}_x \neq 0 \neq \hat{E}_y$. We get a transverse EM waves, that are described by 2×2 system of equations,

$$\begin{pmatrix} S - \mathcal{N}^2 & -iD \\ iD & S - \mathcal{N}^2 \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \end{pmatrix} = 0 \tag{73}$$

In this case of transverse waves, again it is best to introduce the rotating electric field, $\hat{E}^{\pm}$ (see Equation 64) which correspond to a circular polarization of the wave. This enable to decouple the two equations (simply write them explicitly, multiply the second equation by i and add / subtract):

$$\begin{aligned} \left(S - D - \mathcal{N}^2\right) \hat{E}^{+} &= 0, \\ \left(S + D - \mathcal{N}^2\right) \hat{E}^{-} &= 0. \end{aligned} \tag{74}$$

When $\hat{E}^{+} \neq 0$ and $\hat{E}^{-} = 0$, we have a **left-handed circularly polarized wave** (in short: L-wave), with a refractive index $\mathcal{N}_L = \sqrt{S - D}$.

When $\hat{E}^{+} = 0$ and $\hat{E}^{-} \neq 0$, we have a **right-handed circularly polarized wave** (in short: R-wave or R-mode), with a refractive index $\mathcal{N}_R = \sqrt{S + D}$. With the definitions of S and D (Equation 70), we have

$$\begin{aligned} \mathcal{N}_R &= \left(1 - \frac{\omega_{p,e}^2}{\omega(\omega - \omega_{c,e})} - \frac{\omega_{p,i}^2}{\omega(\omega + \omega_{c,i})}\right)^{1/2}, \\ \mathcal{N}_L &= \left(1 - \frac{\omega_{p,e}^2}{\omega(\omega + \omega_{c,e})} - \frac{\omega_{p,i}^2}{\omega(\omega - \omega_{c,i})}\right)^{1/2}. \end{aligned} \tag{75}$$

For $\omega = \omega_{c,e}$, the refractive index of the R-mode becomes $\mathcal{N}_R \to \infty$. The R-mode is said to have a **resonance** with the electron cyclotron frequency.

We can easily understand this resonance by looking at the rotation of the electric field, as is presented in Figure 8. In the rotating frame of reference, the electron experiences a DC electric field, and can therefore gain energy indefinitely.

Similarly, the L-mode has a resonance at the ion cyclotron frequency.

Plotting the refractive index as a function of frequency visualize the regimes in which the R- and L-modes can propagate, and where there will be cutoffs. This is seen in Figure 9, where an artificial ratio of $m_e/m_i = 0.4$ is used. Above the respective cyclotron frequencies,

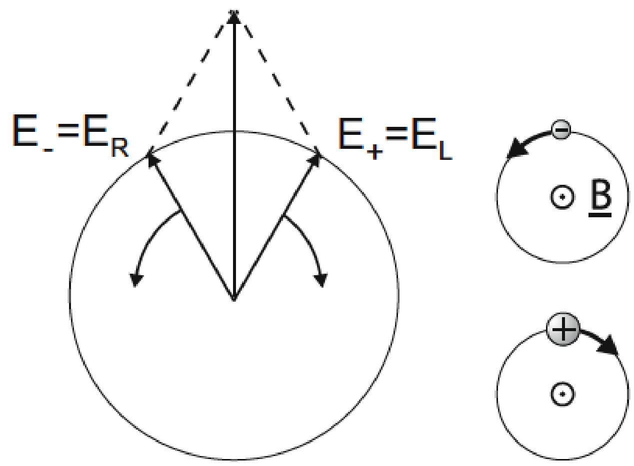

Fig. 8.— Decomposition of a linearly polarized wave into a pair of circularly polarized waves, and comparison of the L- and R-mode rotation with the gyromotion of the electrons and ions. Figure taken from Ref. [2]

the L- and R-waves are in cutoff, $\mathcal{N}^2 < 0$ (shaded region for the R-mode). At high enough frequencies, $\mathcal{N}^2 > 0$ and the modes can propagate. At very high frequencies, the refractive indices of both the L- and R-waves approach the asymptotic value $\mathcal{N} \to 1$ (similar to vacuum).

6.2.1. Experimental consequence: Whistler waves

Early investigators of radio emissions from the ionosphere, encountered various whistling sounds in the audio-frequency range, at 1- 30 kHz. There is typically a series of descending glide tones, which can be heard over a loudspeaker.

We can easily explain them by means of propagation of the R-modes, which is the only propagating mode at the frequency range between $\omega_{c,i} < \omega < \omega_{c,e}$, in the case of $\omega_{p,e}^2 \gg \omega_{c,e}^2$, as is seen in Figure 9.

The origin of these waves is in lightning flashes, which (for an observer in the northern hemisphere) occurs in the southern hemisphere. The lightnings generate radio noise at all

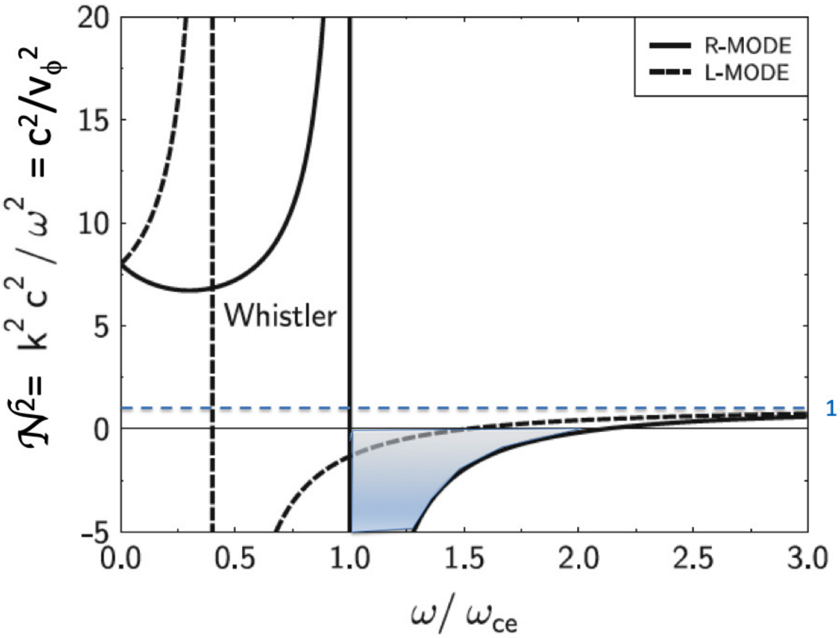

Fig. 9.— The square of the refractive index for wave propagation along the magnetic field, $\mathbf{k} \parallel \mathbf{B}$ as a function of frequency. An artificial mass ratio $m_e/m_i = 0.4$ is assumed. The R-mode has a resonance $\mathcal{N}_R \to \infty$ at $\omega_{c,e}$, while the L-mode has a resonance at the (lower) ion cyclotron frequency, $\omega_{c,i}$. In the high density limit, $\omega_{p,e} \gg \omega_{c,e}$ (as is considered here), only R-waves can propagate in between the ion and electron cyclotron frequencies, while the L-wave has a cutoff, $\mathcal{N}_L^2 < 0$. The cutoff of the R-wave is shown by the shaded region. As ω increases, $\mathcal{N}_{L,R} \to 1$. Figure modified from Ref. [2]

frequencies, including R waves that travel along the earth's magnetic field (see Figure 10).

Both the phase and the group velocities increase with frequency; for $\omega^2 \ll \omega_{c,e}^2 \ll \omega_{p,e}^2$, we have $\mathcal{N}_R \approx \frac{\omega_{p,e}}{\sqrt{\omega\omega_{c,e}}}$, and therefore low frequencies arrive later giving rise to the "chew" sound.

Several whistles can be produced by a single lightning flash, because of propagation along different tubes.

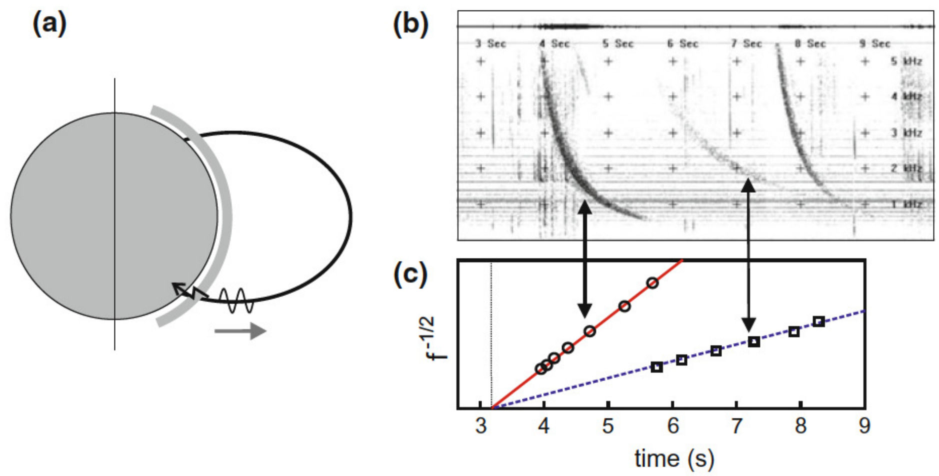

Fig. 10.— **a** A lightning between cloud and ionosphere in the southern hemisphere triggers a whistler wave that travels along the magnetic field. **b** Sonogram of whistler wave events in the northern hemisphere. The strong whistler starting at 4 s is followed by a weak echo of much larger dispersion. **c** The evaluation shows that the first is a one-hop Whistler and the second a three-hop whistler. Both signals follow a $f^{-1/2}$ law, which extrapolates to a common starting point. Figure taken from Ref. [2]

6.2.2. Faraday rotation

The small differences between $\mathcal{N}_R$ and $\mathcal{N}_L$ at high frequencies, namely that the R-waves travel slightly faster, gives rise to a rotation of the plane of polarization of the traveling electromagnetic wave as it propagates along the magnetic field line. This rotation is known as **Faraday rotation**.

A linearly-polarized transverse wave propagating along the magnetic field can be decomposed into a pair of R- and L-mode (see Figure 8). Consider a field that is initially along the $\hat{x}$ direction, $\hat{\mathbf{E}} = \hat{E}_x$. Using Equation 64, we have $\hat{E}^{\pm}(z=0,t=0) = \hat{E}_x(z=0,t=0)$. As the wave propagates along the magnetic field, we have $\hat{E}^{\pm}(z,t) = \hat{E}_x(z=0,t=0)\exp(i(k_\pm z - \omega t))$, where $k_\pm$ are the wavenumbers of the R- and L-modes.

We may write $k_\pm = k_0 \pm \delta k$, where $k_0 = (k_+ + k_-)/2$, $\delta k = (k_+ - k_-)/2$. We then have

$$\begin{aligned} \hat{E}_x(z,t) &= \left(\frac{\hat{E}^+(z,t)+\hat{E}^-(z,t)}{2}\right) = \hat{E}_x(z=0,t=0)e^{i(k_0 z-\omega t)}\cos(\delta k z) \\ \hat{E}_y(z,t) &= \left(\frac{\hat{E}^+(z,t)-\hat{E}^-(z,t)}{2i}\right) = \hat{E}_x(z=0,t=0)e^{i(k_0 z-\omega t)}\sin(\delta k z) \end{aligned} \tag{76}$$

We thus find that the plane of polarization, which, at $z = 0$ was aligned along the $\hat{x}$

axis, is rotating at a rate $\alpha(z) = \delta kz$ about the magnetic field direction (the $\hat{z}$-axis).

In the high-frequency limit, $\omega \gg \{\omega_{p,e}, \omega_{c,e}\}$, we have from Equation 75,

$$\mathcal{N}_{R,L} \equiv \mathcal{N}_{\mp} \approx 1 - \frac{\omega_{p,e}^2}{2\omega(\omega \mp \omega_{c,e})} \tag{77}$$

Using $\mathcal{N} = kc/\omega$, we have

$$\alpha(z = L) \approx \frac{\omega_{p,e}^2 \omega_{c,e} L}{c\omega^2} \propto nBL \tag{78}$$

where the proportionality comes from $n \propto \omega_{p,e}^2$ and $B \propto \omega_{c,e}$.

Thus, measuring the polarization direction for different frequencies, provides a direct measurement of the quantity nBL; thus us a standard techniques used in studying the magnetic fields in galaxies.

6.3. Propagation across the magnetic field

Let us now consider propagation of waves across the magnetic field: $\mathbf{k} \perp \mathbf{B}_0$ or $\psi = \pi/2$ (or $\mathbf{k} = k\hat{x}$).

In this case, the electric field vector may be aligned along the static magnetic field, $\mathbf{E} \parallel \mathbf{B}_0$, or perpendicular to it. When $\mathbf{E} \parallel \mathbf{B}_0$, the wave is called **ordinary mode**, or O-mode. In this mode, the electron and ion motion are along the magnetic field, and therefore the refractive index of the O-mode is not affected by the presence of the **B**-field (hence the name "ordinary mode").

If $\mathbf{E} \perp \mathbf{B}_0$, the wave is called **extraordinary mode**, or X-mode. Using Equation 71, it is described by a 2×2 system of equations,

$$\begin{pmatrix} S & -iD \\ iD & S - \mathcal{N}^2 \end{pmatrix} \cdot \begin{pmatrix} \hat{E}_x \\ \hat{E}_y \end{pmatrix} = 0 \tag{79}$$

As usual, non-vanishing solutions for **E** are found when the determinant of the matrix becomes 0, giving a referactive index

$$\mathcal{N}_X = \left(\frac{S^2 - D^2}{S} \right)^{1/2}. \tag{80}$$

A **resonance** occurs when the index of refraction becomes infinite (namely, the wavelength becomes 0, since $\mathcal{N} = kc/\omega$). For X-modes, resonance appears when $S = 0$. For very high frequencies, we can neglect the ion contribution in Equation 70, and write

$$\omega_{uh} = \left(\omega_{c,e}^2 + \omega_{p,e}^2\right)^{1/2} \tag{81}$$

This is known as the **upper hybrid resonance** ("hybrid" due to the fact that it combines both the cyclotron and plasma frequencies).

A second zero of S may be found close to the ion cyclotron frequency, by using the fact that $\omega_{p,e} \gg \omega_{p,i}$ and $\omega_{c,e} \gg \omega_{c,i}$ (both correct due to $m_e \ll m_i$). One finds

$$\omega_{lh} = \left(\omega_{c,i}^2 + \omega_{p,i}^2 \frac{\omega_{c,e}^2}{\omega_{p,e}^2 + \omega_{c,e}^2}\right)^{1/2} \tag{82}$$

which is known as the **lower hybrid resonance**. In the limit of high electron density, $\omega_{p,e}^2 \gg \omega_{c,e}^2$, the lower hybrid frequency approaches $\omega_{lh} \approx (\omega_{c,i}\omega_{c,e})^{1/2}$ (the second term in Equation 82 becomes dominant). The behavior of the refractive index for the X- and O-modes as a function of the wave frequency is shown in Figure 11.

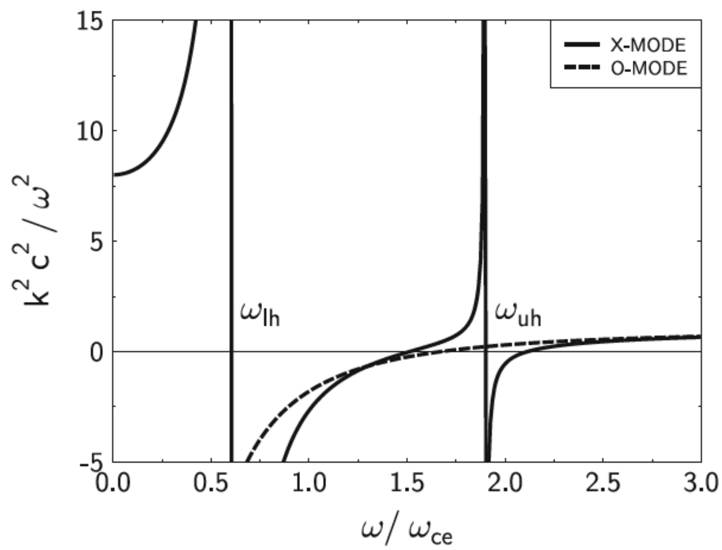

Fig. 11.— The square of the refractive index for waves propagating perpendicular to the magnetic field, as a function of frequency. An artificial mass ratio $m_e/m_i = 0.4$ is chosen. The X-mode has resonances at the lower and upper hybrid frequencies, $\omega_{l,h}$, $\omega_{u,h}$. Figure taken from Ref. [2]

7. Summary: elementary plasma waves

Let us summarize the elementary plasma waves in the following table.

Mode	Electron wave	Ion waves
Electrostatic waves: $\mathbf{k} \parallel \mathbf{E}$, or $B_0 = 0$		
$\mathbf{B}_0 = 0$ or $\mathbf{k} \parallel \mathbf{B}_0$	Electron acoustic waves: (Langmoir waves / plasma oscillations) $\omega^2 = \omega_{p,e}^2 + \frac{3}{2}k^2 v_{th}^2$	Ion acoustic waves: $\omega^2 = k^2 \left(\frac{\hat{\gamma} k_B T_i}{m_i} + \frac{\omega_{p,i}^2 \lambda_D^2}{1+k^2\lambda_D^2} \right)$
$\mathbf{k} \perp \mathbf{B}_0$	upper hybrid resonance $\omega_{uh}^2 = \omega_{c,e}^2 + \omega_{p,e}^2$	lower hybrid resonance $\omega_{lh}^2 \approx \omega_{c,i}\omega_{c,e}$
Electromagnetic waves		
$\mathbf{B}_0 = 0$	Electromagnetic (light) waves: $\omega^2 = \omega_{p,e}^2 + k^2 c^2$	None
$\mathbf{k} \parallel \mathbf{B}_0$	R- (whistler) and L-modes $\mathcal{N}_{R,L}^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{p,e}^2}{\omega(\omega \mp \omega_{c,e})}$	Alfvén waves, $\omega^2 = k^2 v_A^2$
$\mathbf{k} \perp \mathbf{B}_0$	$\mathbf{E}_1 \parallel \mathbf{B}_0$: O mode $\omega^2 = \omega_{p,e}^2 + k^2 c^2$ $\mathbf{E}_1 \perp \mathbf{B}_0$: X mode (Eq. 80) $\mathcal{N}_X^2 = \frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_{p,e}^2}{\omega^2} \frac{\omega^2 - \omega_{p,e}^2}{\omega^2 - \omega_{p,e}^2 - \omega_{c,e}^2}$	Magnetosonic wave (didnt discuss) $\frac{\omega^2}{k^2} = c^2 \frac{v_s^2 + v_A^2}{c^2 + v_A^2}$

Note that electrostatic (longitudinal) waves do not exit in vacuum; they cannot be obtained from Maxwell's equations in vacuum. However, they do exist in plasma, and, being longitudinal, they are similar in nature to sound waves.

REFERENCES

[1] F. Chen, *Introduction to Plasma Physics and Controlled Fusion* (Springer), chapter 3.

[2] A. Piel, *Plasma Physics: An Introduction to Laboratory, Space, and Fusion Plasmas* (Springer), chapter 5.