Kinetic theory

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This part of the course is based on Refs. [1] and [2].

1. Introduction

The fluid theory we used provides a simple description of the plasma. In this model, we considered average behavior of particles filling a small volume in space. We used quantities such as density or (average) velocity. By doing so, we treated the fluid properties as functions of space and time, $(t, \vec{x})$. We could do that, because we had an **underlying assumption**: that the velocity distribution of the particles is **Maxwellian everywhere**. Therefore, we could uniquely specify the particle's distribution using only one number- the temperature, T.

However, clearly, this is not always the case. In particular, at high temperatures, collisions can be rare, and therefore deviations from thermal equilibrium can last for relatively long time. The classical fluid theory does not treat non-Maxwellian velocity distributions; for that, we need to refine the theory. This is done by means of **Vlasov equation**. In Figure 1, a sketch of the different plasma regimes is presented.

2. The Vlasov model

2.1. The distribution function

We would like to add a description of the velocity distribution of particles. When we discussed fluid theory, rather than looking at single particle trajectories, we shifted to a statistical description of the mean properties of plasma at a given point in space. This includes quantities such as the density, $n(\mathbf{r}, t)$ or the flow velocity $\mathbf{u}(\mathbf{r}, t)$.

The total number of particles within a volume element $\Delta V = \Delta x \Delta y \Delta z$ around $\mathbf{r}$ at time t is given by

$$\Delta N = n(\mathbf{r}, t)\Delta x \Delta y \Delta z$$

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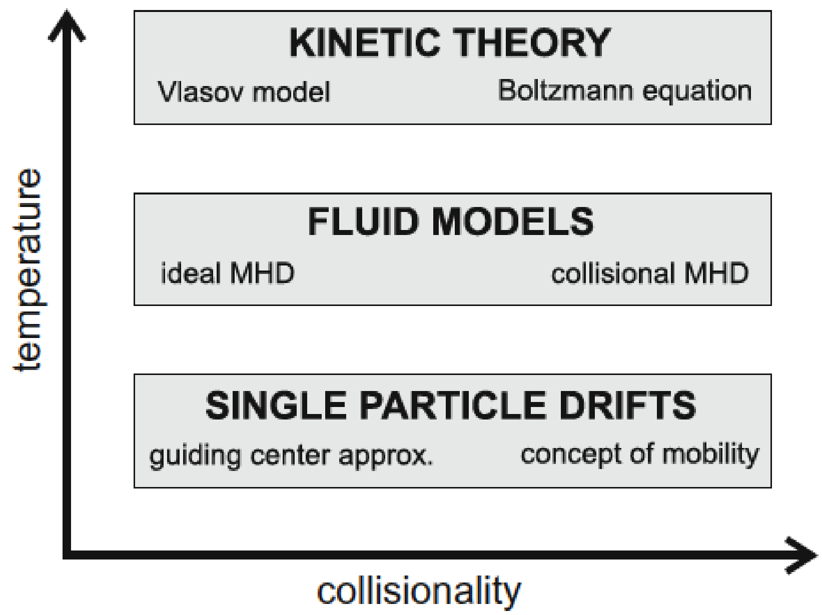

Fig. 1.— The hierarchy of plasma models. Figure taken from Ref. [2]

In kinetic theory, we now look at the evolution of particles not only in (spatial) space, but also in velocity space. In exactly analogue way to the definition of the density, n, we define the **distribution function**, $f(\mathbf{r}, \mathbf{v}, t)$ as the number density of particles (=number of particles per unit volume) per unit velocity $\mathbf{v}..\mathbf{v} + \mathbf{\Delta v}$, namely

$$\Delta N = f(\mathbf{r}, \mathbf{v}, t)\Delta x \Delta y \Delta z \Delta v_x \Delta v_y \Delta v_z. \tag{1}$$

The distribution function f is thus a function of 7 independent variables (3 spatial, 3 velocity and time, or 6-d phase-space and 1 time variable).

Clerly, the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ contains all the information needed to describe the plasma in the fluid model. For example, the density is given by

$$n(\mathbf{r}, t) = \int_{-\infty}^{+\infty} d\mathbf{v} f(\mathbf{r}, \mathbf{v}, t) = \int_{-\infty}^{+\infty} dv_x \int_{-\infty}^{+\infty} dv_y \int_{-\infty}^{+\infty} dv_z f(\mathbf{r}, \mathbf{v}, t) \tag{2}$$

and the average velocity is

$$\langle \mathbf{v}(\mathbf{r}, t) \rangle = \int_{-\infty}^{+\infty} d\mathbf{v}\mathbf{v} \hat{f}(\mathbf{r}, \mathbf{v}, t) \tag{3}$$

where $\hat{f}$ is the normalized distribution function,

$$f(\mathbf{r}, \mathbf{v}, t) = n(\mathbf{r}, t)\hat{f}(\mathbf{r}, \mathbf{v}, t)$$

The same is true for the average of **any** physical quantity,

$$\langle A \rangle = \int_{-\infty}^{+\infty} d\mathbf{v} A \hat{f}(\mathbf{r}, \mathbf{v}, t)$$

2.2. Equtions of kinetic theory: the Vlasov equation

We seek an equation of motion for the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ that generalizes the continuity equation, $\partial n(\mathbf{r}, t)/\partial t + \vec{\nabla}[n(\mathbf{r}, t)\mathbf{u}(\mathbf{r}, t)] = 0$. This is easily explained if we look at only 1-d system, which has only the coordinates (x, v_x). Similar to the argument that led to the continuity equation, we argue that the particle balance within a phase space volume $\Delta x \Delta v_x$ is determined by (i) the difference of inflow and outflow in real space; and (ii) the acceleration and deceleration that changes the flow velocity. In addition, the change in the distribution function **within** the box may exist, due to collisions within the box. Thus, in analogy to the continuity equation, we have

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(f v_x) + \frac{\partial}{\partial v_x}(f a_x) = \left(\frac{\partial f}{\partial t}\right)_c \tag{4}$$

Here, $f v_x$ is the flux in real space, and $f a_x$ is the flux in v_x direction caused by an acceleration a_x. $\left(\frac{\partial f}{\partial t}\right)_c$ is the time rate of change in the distribution function due to collisions.

When the phase-space coordinate v_x is independent of x, and the x-component of the Lorentz force F_x is independent of v_x, we can take these quantities out of the derivatives. Generalizing to 3d, we have

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \vec{\nabla} f + \frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_c . \tag{5}$$

Equation 4 is known as **Boltzmann Equation**. The vector derivative is defined by $\frac{\partial}{\partial \mathbf{v}} \equiv \left(\frac{\partial}{\partial v_x}, \frac{\partial}{\partial v_y}, \frac{\partial}{\partial v_z}\right)$.

We now make two simplified assumptions: (i) At sufficiently high temperatures, the collisions can be neglected; and (ii) The particle acceleration, a, is determined by the electric and magnetic fields,

$$\mathbf{a} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{6}$$

These fields are the sum of external fields and internal fields originating from the particles currents. Using these assumptions in Boltzmann's equation, we obtain the **Vlasov equation**,

$$\boxed{\frac{\partial f}{\partial t} + \mathbf{v} \cdot \vec{\nabla} f + \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0.} \tag{7}$$

There are individual equations for electrons and ions. [2]

2.3. Properties of the Vlasov Equation

Let us first discuss a few important properties of Vlasov equation.

1. The Vlasov equation conserves the total number of particles, N of a species. Let us prove it in the 1-d case:

$$\begin{aligned} \frac{\partial N}{\partial t} &= \frac{\partial}{\partial t} \int \int f dx dv \\ &= -\int \int v \frac{\partial f}{\partial x} dx dv - \int \int a \frac{\partial f}{\partial v} dx dv \\ &= -\int_{-\infty}^{+\infty} dv \left\{ [vf]_{x=-\infty}^{x=+\infty} - \int_{-\infty}^{+\infty} f \frac{dv}{dx} dx \right\} \\ &\quad - \int_{-\infty}^{+\infty} dx \left\{ [af]_{v=-\infty}^{v=+\infty} - \int_{-\infty}^{+\infty} f \frac{da}{dv} dv \right\} = 0 \end{aligned} \tag{8}$$

   In the last line, we used the fact that f vanishes for $x \to \pm\infty$, otherwise the total number of particles would be infinite (for 3d, it must decay faster than r^{-2}). Similarly, f decays faster than v^{-2} for $v \to \pm\infty$, otherwise the total kinetic energy, $E_k \propto \int v^2 f dv dx$ would become infinite.

   Furthermore, $dv/dx = 0$ because v and x are independent variables, and $da/dv = 0$ because the x-component of the Lorentz force does not depend on v_x.

2. Any function g which can be written in terms of the total energy of the particles, e.g., $g\left[\frac{1}{2}mv^2 + q\Phi(x)\right]$ is a solution of the Vlasov Equation, written in the form

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{q}{m}\frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial v} = 0.$$

3. The phase-space density (= the distribution function) f is constant along the trajectory of a test particle that moves under the influence of the electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$.

[2] When Coulomb collisions are considered, the equation is known as Fokker-Planck equation.

Proof: Let $[\mathbf{x}(t), \mathbf{v}(t)]$ be the coordinates along the trajectory of a particle that follows from the equation of motion, $m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, and let $\dot{\mathbf{x}} = \mathbf{v}$. Then:

$$\begin{aligned} \frac{df(\mathbf{x}(t),\mathbf{v}(t),t)}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \frac{d\mathbf{x}}{dt} + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{d\mathbf{v}}{dt} \\ &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}} \cdot \mathbf{v} + \frac{\partial f}{\partial \mathbf{v}} \cdot \frac{q}{m} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) = 0 \end{aligned} \tag{9}$$

4. Vlasov equation is invariant under time reversal $(t \to -t)$, $(\mathbf{v} \to -\mathbf{v})$. This also implies that the entropy is conserved: for non-equilibrium system, the entropy is defined by

$$S = -\int d^3x d^3v f \ln f$$

and is also conserved, $dS/dt = 0$, in a similar way to the calculaiton in item (1) above.

2.4. Relation between the Vlasov equation and fluid models

The Vlasov model (or kinetic theory) is more comprehensive than the fluid model, as now arbitrary distribution functions can be treated. Thus, the fluid model is in fact no more than a special case of the Vlasov model.

2.4.1. Zeroth order moment: the continuity equation

The fluid equations that we have been using, are simply moments of the Boltzmann equation. For simplicity, in the calculation below we limit the discussion to Vlasov equation. The lowest moment (**zeroth-order moment**) is obtained by integrating Vlasov equation (7) over the velocity:

$$\int \frac{\partial f}{\partial t} d\mathbf{v} + \int \mathbf{v} \cdot \vec{\nabla} f d\mathbf{v} + \frac{q}{m} \int (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = 0. \tag{10}$$

Since $\mathbf{v}$ is independent variable, we can write the first term:

$$\int \frac{\partial f}{\partial t} d\mathbf{v} = \frac{\partial}{\partial t} \int f d\mathbf{v} = \frac{\partial n}{\partial t} \tag{11}$$

Similarly, since $\mathbf{v}$ is independent on $\mathbf{x}$, it is not affected by the operator $\vec{\nabla}$. Hence, we can write the second term as

$$\int \mathbf{v} \cdot \vec{\nabla} f d\mathbf{v} = \vec{\nabla} \cdot \int \mathbf{v} f d\mathbf{v} = \vec{\nabla} \cdot (n\langle \mathbf{v} \rangle) \equiv \vec{\nabla} \cdot (n\mathbf{u}) \tag{12}$$

Here, the average velocity $\mathbf{u}$ is, by definition, the fluid velocity.

The next two terms vanish:

$$\int \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = \int \frac{\partial}{\partial \mathbf{v}} \cdot (f\mathbf{E}) d\mathbf{v} = \int_{S_\infty} f\mathbf{E} \cdot d\mathbf{S} = 0 \tag{13}$$

where the perfect divergence is integrated to give the value of $f\mathbf{E}$ on the surface at $v = \infty$. This vanishes, as $f \to 0$ faster than v^{-2} as $v \to \infty$, to ensure convergence of energy, as explained above.

Finally, the $\mathbf{v} \times \mathbf{B}$ term also vanishes, as

$$\int (\mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = \int \frac{\partial}{\partial \mathbf{v}} \cdot (f\mathbf{v} \times \mathbf{B}) d\mathbf{v} - \int f \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{v} \times \mathbf{B}) d\mathbf{v} = 0 \tag{14}$$

The first integral can again be converted to a surface integral, which vanishes when f drops faster than $\mathbf{v}$ when $\mathbf{v} \to \infty$; e.g., for a Maxwellian distribution. The second integral vanishes since $(\mathbf{v} \times \mathbf{B})$ is perpendicular to $(\partial/\partial \mathbf{v})$.

Combining this all together, we obtain from Equation 10

$$\frac{\partial n}{\partial t} + \vec{\nabla}(n\mathbf{u}) = 0 \tag{15}$$

which is the continuity equation we derived above.

Note that if we were to take the full Boltzmann equation (5) with the collision term on the right hand side, the result will not change. For on the right hand side we would have

$$\int \left(\frac{\partial f}{\partial t}\right)_c d\mathbf{v} = \frac{\partial}{\partial t}\left[\int f d\mathbf{v}\right] = 0, \tag{16}$$

since collisions do not change the total number of particles (we are neglecting creation or annihilation of particles).

2.4.2. First order moment: momentum transport equation

The next moment of Vlasov equation is obtained by multiplying Vlasov equation (7) by $m\mathbf{v}$, and integrating over $d\mathbf{v}$.

$$m \int \mathbf{v} \frac{\partial f}{\partial t} d\mathbf{v} + m \int \mathbf{v} \left(\mathbf{v} \cdot \vec{\nabla}\right) f d\mathbf{v} + q \int \mathbf{v} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = 0. \tag{17}$$

The first term in Equation 17 is

$$m \int \mathbf{v} \frac{\partial f}{\partial t} d\mathbf{v} = m \frac{\partial}{\partial t} \int \mathbf{v} f d\mathbf{v} \equiv m \frac{\partial}{\partial t} (n\mathbf{u}) \tag{18}$$

where in the first equality we used the fact that $\mathbf{v}$ is an independent parameter to take the time derivative out of the integration.

We write the third integral as

$$\begin{aligned} \int \mathbf{v} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \tfrac{\partial f}{\partial \mathbf{v}} d\mathbf{v} \;\; = & \;\; \int \tfrac{\partial}{\partial \mathbf{v}} \cdot \left[f \mathbf{v} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)\right] d\mathbf{v} \\ & - \int f \mathbf{v} \tfrac{\partial}{\partial \mathbf{v}} \cdot \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) d\mathbf{v} - \int f \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \tfrac{\partial}{\partial \mathbf{v}} \mathbf{v} d\mathbf{v} \end{aligned} \tag{19}$$

The first two integrals on the right hand side vanish, from the same reasons as before - $f \to 0$ faster than v^{-2} as $v \to \infty$, to ensure conservation of energy.

As for the third term, we note that $\frac{\partial}{\partial \mathbf{v}} \mathbf{v}$ is just the identity tensor, $\bar{\mathbf{I}}$. Overall, we thus get

$$q \int \mathbf{v} \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) \cdot \frac{\partial f}{\partial \mathbf{v}} d\mathbf{v} = -q \int f \left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) d\mathbf{v} = -qn \left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) \tag{20}$$

Finally, to evaluate the second integral in Equation 17, we take the following steps. First, we use the fact that $\mathbf{v}$ is an independent variable, and therefore does not depend on the gradient operator, $\vec{\nabla}$, to write

$$\int \mathbf{v} \left(\mathbf{v} \cdot \vec{\nabla}\right) f d\mathbf{v} = \int \vec{\nabla} \cdot (f \mathbf{v}\mathbf{v}) d\mathbf{v} = \vec{\nabla} \cdot \int f \mathbf{v}\mathbf{v} d\mathbf{v} \tag{21}$$

The average of a quantity $\langle x \rangle$ is its weighted integral over $\mathbf{v}$, multiplied by $1/n$ for normalization (see Equation 3; $\langle x \rangle = \frac{\int f x d\mathbf{v}}{n}$). We thus have

$$\vec{\nabla} \cdot \int f \mathbf{v}\mathbf{v} d\mathbf{v} = \vec{\nabla} \cdot (n \langle \mathbf{v}\mathbf{v} \rangle) \tag{22}$$

Next, we divide the velocity $\mathbf{v}$ into the average (fluid) velocity $\mathbf{u}$, and a thermal (random) velocity, $\mathbf{v}_{th}$:

$$\mathbf{v} = \mathbf{u} + \mathbf{v}_{th}$$

Since $\mathbf{u}$ is already an average, we have

$$\vec{\nabla} \cdot (n \langle \mathbf{v}\mathbf{v} \rangle) = \vec{\nabla} \cdot (n\mathbf{u}\mathbf{u}) + \vec{\nabla} \cdot (n \langle \mathbf{v}_{th} \mathbf{v}_{th} \rangle) + 2\vec{\nabla} \cdot (n\mathbf{u} \langle \mathbf{v}_{th} \rangle) \tag{23}$$

The average of $\langle \mathbf{v}_{th} \rangle = 0$. The quantity $mn \langle \mathbf{v}_{th} \mathbf{v}_{th} \rangle$ is just the shear stress (see "Fluids", equation 27):

$$\bar{\mathbf{P}} = mn \langle \mathbf{v}_{th} \mathbf{v}_{th} \rangle. \tag{24}$$

Note that in 1-d Maxwellian distribution we have $m\langle v_{th} v_{th}\rangle = k_B T$ and the thermal gas pressure is simply $p = n k_B T$.

The first term on the right hand side of Equation 23 is

$$\vec{\nabla} \cdot (n\mathbf{u}\mathbf{u}) = \mathbf{u}\vec{\nabla} \cdot (n\mathbf{u}) + n(\mathbf{u} \cdot \vec{\nabla})\mathbf{u} \tag{25}$$

Collecting terms, we finally write Equation 17 using Equations 18, 25, 24 and 20 as

$$m\frac{\partial}{\partial t}(n\mathbf{u}) + m\mathbf{u}\vec{\nabla} \cdot (n\mathbf{u}) + mn(\mathbf{u} \cdot \vec{\nabla})\mathbf{u} + \vec{\nabla} \cdot \bar{\mathbf{P}} - qn\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) = 0 \tag{26}$$

We now combine the first two terms, using the continuity equation 15, to write

$$m\frac{\partial}{\partial t}(n\mathbf{u}) + m\mathbf{u}\vec{\nabla} \cdot (n\mathbf{u}) = mn\frac{\partial}{\partial t}(\mathbf{u})$$

and Equation 26 becomes

$$mn\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \vec{\nabla})\mathbf{u}\right] = -\vec{\nabla} \cdot \bar{\mathbf{P}} + qn\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right) \tag{27}$$

which is the momentum transport equation, which we derived in "Fluids" (Equation 26). This equation describes the flow of momentum.

2.4.3. Second order moment: the flow of energy

We may proceed in the same way: if we want to describe the flow of energy, we may take the next moment of the Vlasov (or Boltzmann, if collisions are included) by multiplying the Vlasov equation by $\frac{1}{2}m\mathbf{v}^2$, and integrate. If we were to do this, we would have obtained the **heat flow equation**.

Rather than working hard, this equation can be replaced by a much simpler equation - the equation of state, $p \propto \rho^{\hat{\gamma}}$, which is in fact a simple form of the heat flow equation for zero thermal conductivity.

3. Plasma oscillations and Landau damping

We now consider small amplitude electrostatic waves in unmagnetized plasmas. The treatment will be done in 1d. This problem has been treated when we dealt with waves. However, here we will see kinetic effects that we could not see before, in particular the phenomenon of **Landau damping**: damping of the waves via interaction with the plasma particles, despite the fact that the plasma is collisionless.

3.1. Electrostatic electron waves

Similar to the treatment we did in "waves", we look for electron oscillations near the electron plasma frequency. The ions do not participate in the wave motion, but only serve to neutralize the background. The treatment here will be 1-d.

We split the electron distribution function $f_e(x, v, t)$ into two parts: a homogeneous (=space-independent) and stationary (=time independent) part, $f_{e,0}(v)$, and a small perturbation, $f_{e,1}(x, v, t)$:

$$f_e(x, v, t) = f_{e,0}(v) + f_{e,1}(x, v, t). \tag{28}$$

We assume that $f_{e,0}(v)$ is a Maxwellian, while $f_{e,1}(x, v, t)$ is a small, wave-like perturbation:

$$f_{e,0}(v) = n_{e,0} \left(\frac{m_e}{2\pi k_B T} \right)^{1/2} \exp\left[-\frac{m_e v^2}{2k_B T} \right] \tag{29}$$

where $T \equiv T_e$ is the electron temperature, and

$$f_{e,1} = \hat{f}_{e,1} \exp[i(kx - \omega t)]. \tag{30}$$

Linearizing Vlasov Equation (7), using the wave representation (eq. 30), we get

$$\frac{\partial f_{e,1}}{\partial t} + v \frac{\partial f_{e,1}}{\partial x} - \frac{q}{m_e} E_1 \frac{\partial f_{e,0}}{\partial v} = 0, \tag{31}$$

and

$$-i\omega \hat{f}_{e,1} + ikv \hat{f}_{e,1} - \frac{q}{m_e} \hat{E}_1 \frac{\partial f_{e,0}}{\partial v} = 0. \tag{32}$$

from which we get the perturbed electron distribution function,

$$\hat{f}_{e,1} = i \frac{q}{m} \frac{(\partial f_{e,0}/\partial v)}{\omega - kv} \hat{E}_1 \tag{33}$$

As we see, for $v = \omega/k$, the denominator of Equation 33 vanishes. Electrons having $v \approx \omega/k$ are called **resonant particles**.

The relationship between the wave electric field E_1 and the perturbed distribution function $f_{e,1}$ is set by Poisson's equation:

$$\epsilon_0 \vec{\nabla} \cdot \mathbf{E}_1 = -q n_1 = -q \int f_1 d\mathbf{v}. \tag{34}$$

Using the wave representation in 1-d, we have

$$ik\epsilon_0 \hat{E}_1 = -q \int \hat{f}_{e,1} dv = \frac{1}{ik} \frac{\omega_{p,e}^2 \epsilon_0}{n_{e,0}} \hat{E}_1 \int_{-\infty}^{+\infty} \frac{(\partial f_{e,0}/\partial v)}{\omega/k - v} dv \tag{35}$$

where we used the electron plasma frequency, $\omega_{p,e}^2 = n_{e,0}q^2/\epsilon_0 m_e$. We thus obtain the dispersion relation,

$$1 = \frac{\omega_{p,e}^2}{k^2}\frac{1}{n_{e,0}}\int_{-\infty}^{+\infty}\frac{(\partial f_{e,0}/\partial v)}{v-\omega/k}dv. \tag{36}$$

3.1.1. Solution for warm plasma

The singularity at $v = \omega/k$ implies that the integral in Equation 36 is not straightforward to evaluate. However, for cold enough plasma, the thermal speed of the electrons is sufficiently small compared with the phase velocity of the wave, ω/k, that an approximate solution can be (relatively) easy obtained, by neglecting those particles with $v = \omega/k = v_\phi$.

To do that, we note that when $f_{e,0}$ is a Maxwellian (Equation 29), we may write its derivative as

$$\frac{\partial f_{e,0}}{\partial v} = -n_{e,0}2\pi v\left(\frac{m_e}{2\pi k_B T}\right)^{3/2}\exp\left[-\frac{m_e v^2}{2k_B T}\right] = -n_{e,0}\frac{2}{\sqrt{\pi}}\left(\frac{v}{v_{th}^3}\right)\exp\left[-\frac{v^2}{v_{th}^2}\right] \tag{37}$$

where the thermal velocity is $\frac{1}{2}mv_{th}^2 = k_B T$.

We ignore the contribution from the resonant particles, due to the exponentially small factor in the numerator (see Figure 2). The contribution to the dispersion relation 36 is from particles in the interval $[-v_{th}..v_{th}]$. In this regime, we can Taylor expand $(\omega/k - v)^{-1}$ as

$$\frac{1}{\omega/k - v} = \left(\frac{k}{\omega}\right)\left(\frac{1}{1-\frac{vk}{\omega}}\right) \approx \frac{k}{\omega} + v\frac{k^2}{\omega^2} + v^2\frac{k^3}{\omega^3} + ...$$

We may now carry the integral in Equation 36, using the mathematical identity

$$\int_0^\infty x^m e^{-ax^2} = \frac{\Gamma\left(\frac{m+1}{2}\right)}{2a^{\frac{m+1}{2}}}$$

We note that when m is odd ($m = 1, 3, 5, ...$) due to symmetry, $\int_{-\infty}^{+\infty} x^m e^{-ax^2} = 0$, while when m is even, $m = 2n$ ($n = 1, 2, 3, ...$), we have

$$\begin{aligned}\int_{-\infty}^{\infty} x^m e^{-ax^2} = 2\times\frac{\Gamma\left(\frac{2n+1}{2}\right)}{2a^{\frac{2n+1}{2}}} &= \frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{2^n}\sqrt{\pi}\times\frac{1}{a^n\sqrt{a}} \\ &= \frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}{(2a)^n}\left(\frac{\pi}{a}\right)^{1/2}\end{aligned}$$

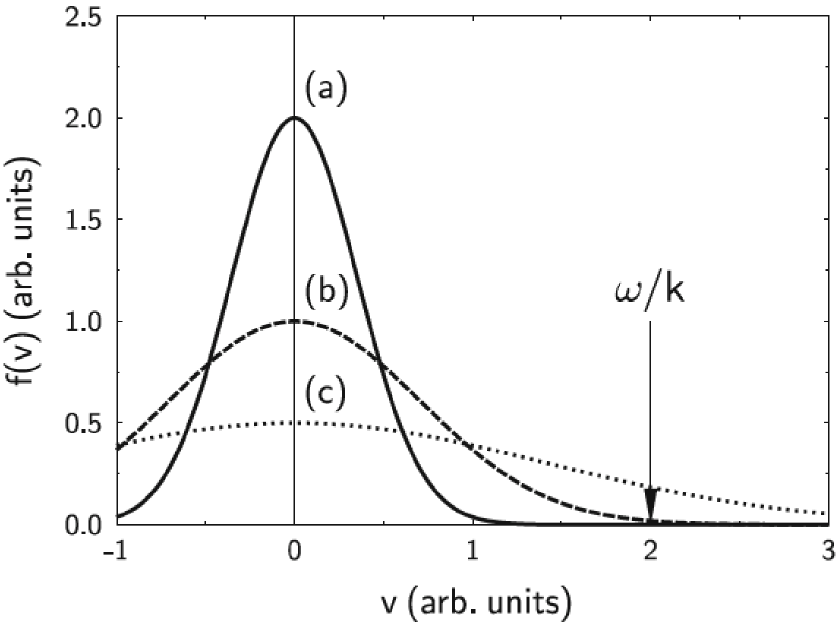

Fig. 2.— Relation between phase velocity and the width of the electron distribution function for (a) cold, (b) warm, and (c) hot plasma. Figure taken from Ref. [2]

We retain terms up to 4th order in the phase velocity, ω/k. Equation 36 becomes:

$$\begin{aligned} 1 &= -\frac{\omega_{p,e}^2}{k^2}\frac{1}{n_{e,0}}\int_{-\infty}^{+\infty}\frac{(\partial f_{e,0}/\partial v)}{\omega/k-v}dv \\ &= +\frac{\omega_{p,e}^2}{k^2}\frac{2}{\sqrt{\pi}v_{th}^3}\int_{-\infty}^{+\infty}\left[\left(\frac{vk}{\omega}\right)+\left(\frac{vk}{\omega}\right)^2+\left(\frac{vk}{\omega}\right)^3+\left(\frac{vk}{\omega}\right)^4+...\right]e^{-\frac{v^2}{v_{th}^2}} \\ &= +\frac{\omega_{p,e}^2}{k^2}\frac{2}{\sqrt{\pi}}\frac{1}{v_{th}^3}\left[\frac{\sqrt{\pi}}{2}\frac{k^2}{\omega^2}v_{th}^3+\frac{3\sqrt{\pi}}{4}\frac{k^4}{\omega^4}v_{th}^5+...\right] \\ &= \frac{\omega_{p,e}^2}{\omega^2}+\frac{3}{2}\frac{\omega_{p,e}^2}{\omega^4}k^2v_{th}^2 \end{aligned} \tag{38}$$

If the thermal correction is small, we may replace $\omega^2 \approx \omega_{p,e}^2$ in the last term. We then obtain the dispersion relation derived previously for the electron acoustic waves (also called "Bohm-Gross waves": see "Waves, equation 48):

$$\omega^2 = \omega_{p,e}^2 + \frac{3}{2}k^2v_{th}^2 = \omega_{p,e}^2 + \hat{\gamma}_e k^2 \frac{k_B T_e}{m_e} \tag{39}$$

where we used $(1/2)m_e v_{th}^2 = k_B T_e$. Note that we did not have to specify the adiabatic index $\hat{\gamma}_e = 3$ for a 1-d adiabatic compression, as we had to in the original derivative. Rather, the adiabaticity of the process followed from the limit $v_{th} \ll \omega/k$, and was obtained from the coefficients of the lowest-order thermal correction.

Note that this derivation is based on the assumption of $v \ll \omega/k$, which is valid for cold, or warm plasma - otherwise the Taylor expansion would not be valid.

3.2. Landau damping

Let us now have a closer look at the integral describing the dispersion relation (Equation 36):

$$1 = \frac{\omega_{p,e}^2}{k^2} \frac{1}{n_{e,0}} \int_{-\infty}^{+\infty} \frac{(\partial f_{e,0}/\partial v)}{v - \omega/k} dv.$$

This integral has a pole at $v = \omega/k$, and thus is improperly defined. Thus, what we are really considering is Cauchy principal value of the integral. [3]

Integrals of the general type

$$\int_{-\infty}^{+\infty} \frac{F(u)}{v - u} dv$$

are evaluated using the residue theorem. This implies that we treat $u \equiv \omega/k$ as a complex phase velocity, and ω a complex frequency. This should not be of a concern - an imaginary ω simply means that the waves are damped (or amplified by instability).

We need to determine the contour of integration in the complex v plane. There are several possible contours, according to Landau, which are named after him, **Landau contour**.

There are several delicate points here. First, we need to take the upper half of the plane (rather than the lower half) since damping of the waves will arise only when taking this half (mathematically, this can be proved by Laplace analysis of the problem, which I save you). Second, we need to ensure that all the poles lie on the same side of the contour - they will all go **above** the contour (see Figure 3). Finally, we note that the Landau contour is **not** closed from above, but rather just goes from $v = -\infty$ to $+\infty$, because the function $\partial f_e/\partial v$ is **not** necessarily well-behaved there- indeed for a Maxwellian distribution $\exp(-v^2/v_{th}^2)$ is not approaching 0 when $v \to \pm i\infty$.

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[3] Cauchy principal value is a method of assigning values to certain improper integrals. If a function f has a singularity at a finite number b, Cauchy principal value is defined by

$$P \int_a^c f(x)dx = \lim_{\epsilon \to 0^+} \left[\int_a^{b-\epsilon} f(x)dx + \int_{b+\epsilon}^c f(x)dx \right].$$

Here, b is a point such that $\int_a^b f(x)dx = \pm\infty$ for any $a < b$, and $\int_b^c f(x)dx = \mp\infty$ for any $c > b$.

This can be explained as for a general function f, we have, from the residue theorem

$$\int_{C_1} f dv + \int_{C_2} f dv = 2\pi i Res(\omega/k)$$

This can work out provided that the 2nd integral vanish, but this is not the case here.

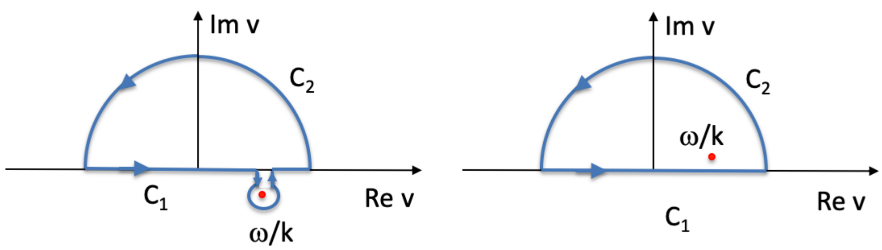

Fig. 3.— Integration contours for Landau problems. (left): $Im(\omega) < 0$; (right): $Im(\omega) > 0$

Rather, the Landau contour goes along the real plane, but with a slight deform below the singularity- see Figure 4

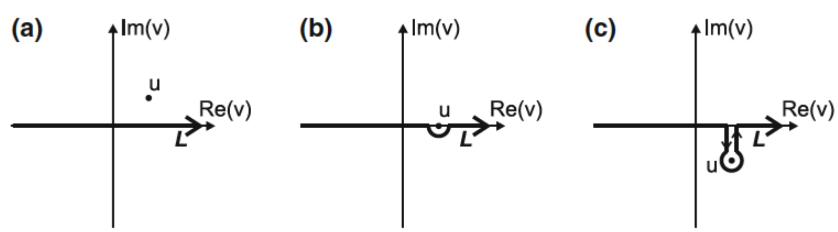

Fig. 4.— (a) The Landau contour L for $Im(u) > 0$ (left), follows the $Re(v)$ axis. (b) for $Im(u) = 0$, the Landau contour passes below the pole. (c) The Landau contour encircles the pole for $Im(u) < 0$. Figure taken from Ref. [2].

In the following, we assume that the imaginary part of u is small compared to the real part. Therefore, in evaluating the integral in equation 36, we use Cauchy principal value, and the contribution from the semi-circle in the Landau contour (Figure 4, part (b).) The later is 1/2 of the residue at the pole.

A. The residue theorem

Calculation of the Landau damping makes use of the **residue theorem**, familiar from complex function analysis.

Let Γ be a positively oriented (i.e. anticlockwise) simple closed contour [on the complex plane] within (and on) which a function $f(z)$ is analytic except for a finite number of singular points $z_1, ..., z_n$ in the interior of Γ. The residue theorem states that

$$\oint_\Gamma f(z)dz = 2\pi i \sum_{i=1}^{n} b_i, \tag{A1}$$

where b_i is the *residue* of $f(z)$ at the singular point z_i. The residue of a function with an isolated singular point z_0 is defined as the coefficient c_1 of the Laurent expansion of $f(z)$ about z_0,

$$f(z) = \sum_{n=0}^{\infty}(z - z_0)^n + \frac{c_1}{z - z_0} + \frac{c_2}{(z - z_0)^2} + ... \tag{A2}$$

for $0 < |z - z_0| < R$, the radius of convergence.

A.1. The use of the residue theorem to calculate integrals

The residue theorem provides us with a strong tool to calculate **improper integrals**, of the form $\int_{-\infty}^{+\infty} f(x)dx$ (where x is real).

Such an integral may diverge at a certain point(s), x_j [which could also be $\pm\infty$].

In order to solve this integral, we look at the integral $\int_\Gamma f(z)dz$, where z is now taken on the complex plane, and Γ is a curve which contains the upper half circle C_R^+, plus the region $-R.. + R$.

According to the residue theorem, we have

$$\int_\Gamma f(z)dz = \int_{C_R^+} f(z)dz + \int_{-R}^{+R} f(x)dx = 2\pi i \sum_{j=1}^{k} res(f, z_j) \tag{A3}$$

where z_j are poles of f in the upper half plane. Thus,

$$\int_{-R}^{+R} f(x)dx = 2\pi i \sum_{j=1}^{k} res(f, z_j) - \int_{C_R^+} f(z)dz \tag{A4}$$

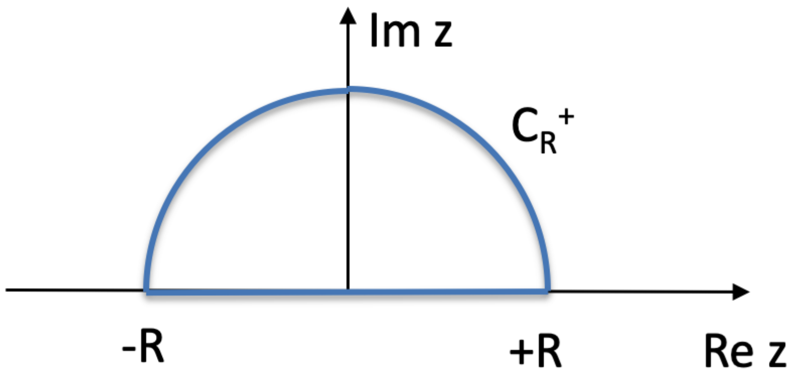

Fig. 5.— Integration contour in the complex plane.

If we can write the function $f(x)$ as a ratio of polynoms, such that

$$|f(x)| < \frac{M}{x^2}$$

where M is some number (this is in fact needed for the improper integral to converge) then when taking the limit $R \to \infty$ we have

$$\left| \int_{C_R^+} f(z) dz \right| < \frac{M}{R^2} \cdot \pi R = \frac{\pi M}{R} \to 0$$

when $R \to \infty$. We are thus left with

$$\int_{-\infty}^{+\infty} f(x) dx = 2\pi i \sum_{j=1}^{k} res(f, z_j) \tag{A5}$$

Example. Calculate the integral $I = \int_{-\infty}^{+\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$.

The function $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$ has poles at $z = +i$, $z = +2i$, on the upper plane.

Calculate:

$$z = \frac{1}{(z-i)} \frac{z^2}{(z+i)(z^2+4)} \quad \to res(z, i) = \frac{-1}{2i \times 3} = \frac{i}{6}$$

and

$$z = \frac{1}{(z-2i)} \frac{z^2}{(z^2+1)(z+2i)} \quad \to res(z, 2i) = \frac{-4}{(-3)(4i)} = -\frac{i}{3}$$

Overall, we get

$$I = 2\pi i \left(\frac{i}{6} - \frac{i}{3} \right) = \frac{\pi}{3}.$$

REFERENCES

[1] F. Chen, *Introduction to Plasma Physics and Controlled Fusion* (Springer), chapter 7.

[2] A. Piel, *Plasma Physics: An Introduction to Laboratory, Space, and Fusion Plasmas* (Springer), chapter 9.