Classical Field Theory

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We begin by discussing various aspects of classical fields. We will cover only the bare minimum ground necessary before turning to the quantum theory, and will return to classical field theory at several later stages in the course when we need to introduce new ideas.

1.   The dynamics of fields

Definition: a **field** is a physical quantity defined at every point in space and time: $(\vec{x}, t)$. While classical particle mechanics deals with a finite number of generalized coordinates - or degrees of freedom, $q_a(t)$ (indexed by a label a), in field theory we are interested in the dynamics of fields

$$\phi_a(\vec{x}, t) \tag{1}$$

where *both* a and $\vec{x}$ are considered as labels. Thus we are dealing with a system with an *infinite number of degrees of freedom* - at least one for each point $\vec{x}$ in space. **Notice that the concept of position has been relegated from a dynamical variable in particle mechanics to a mere label in field theory**.

As opposed to QM, in QFT the dynamical degrees of freedom are the values of ϕ at every single point. As opposed to QM, where X was changed into an operator, in QFT x just stays as a label. Moreover, I stress that ϕ is **not** a wave function; although it obeys certain equations that *look* like the same equations in QM, it has a different interpretation.

Example of a Field (I): The Electromagnetic Field

The most familiar examples of fields from classical physics are the electric and magnetic fields, $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$. Both of these fields are spatial 3-vectors. In a more sophisticated treatment of electromagnetism, we derive these two 3-vectors from a single 4-component field, $A^\mu(\vec{x}, t) \equiv (\phi, \vec{A})$, where $\mu = 0, 1, 2, 3$ shows that this field is a vector in *spacetime*.

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The electric and magnetic fields are given by

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \vec{\nabla} \times \vec{A} \tag{2}$$

which ensure that two of Maxwell's equations, $\vec{\nabla} \cdot \vec{B} = 0$ and $d\vec{B}/dt = -\vec{\nabla} \times \vec{E}$, hold immediately as identities.

1.1. Dynamics of fields: Lagrangian formalism

Next we are interested in knowing the **dynamics** of fields, namely, how they evolve. We could do that in various ways. As an example, for the electromagnetic field, we could write the equations of motion, which are Maxwell's equations. However, we choose a more concise way: **the dynamics of the field is governed by a Lagrangian which is a function of $\phi(\vec{x}, t)$, $\dot{\phi}(\vec{x}, t)$, and $\nabla\phi(\vec{x}, t)$.** In all the systems we study in this course, the Lagrangian is of the form,

$$L(t) = \int d^3x \mathcal{L}(\phi_a, \partial_\mu \phi_a), \tag{3}$$

where $\mathcal{L}$ is officially called **Lagrangian density**, although everyone simply calls it the Lagrangian. The action is

$$S = \int_{t_1}^{t_2} dt \int d^3x \mathcal{L} = \int d^4x \mathcal{L}. \tag{4}$$

Note: Recall that in particle mechanics L depends on q and $\dot{q}$, but not $\ddot{q}$. In field theory we similarly restrict to Lagrangians $\mathcal{L}$ depending on ϕ and $\dot{\phi}$, but not $\ddot{\phi}$. In principle, there is nothing to stop $\mathcal{L}$ depending on $\nabla\phi$, $\nabla^2\phi$, $\nabla^3\phi$, etc. However, with an eye to later Lorentz invariance, we will only consider Lagrangians depending on $\nabla\phi$ and not higher derivatives.

The **equations of motion** can be determined by **the principle of least action**. We vary the path, keeping the end points fixed and require $\delta S = 0$,

$$\begin{aligned} \delta S &= \int d^4x \left[\frac{\partial\mathcal{L}}{\partial\phi_a}\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a) \right] \\ &= \int d^4x \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right) \right]\delta\phi_a + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\delta\phi_a \right) \right\} \end{aligned} \tag{5}$$

The last term is a total derivative and vanishes for any $\delta\phi_a(\vec{x}, t)$ that decays at spatial infinity and obeys $\delta\phi_a(\vec{x}, t_1) = \delta\phi_a(\vec{x}, t_2) = 0$. Note that equation 5 is similar to the one derived for particle mechanics, only now the Lagrangian depends on **both the time derivate ($\dot{\phi}$) and the spatial derivative ($\nabla\phi$) of** ϕ; hence the use of the 4-derivative $\partial_\mu\phi$. Requiring $\delta S = 0$ for all such paths yields the Euler-Lagrange equations of motion for the fields ϕ_a,

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)} \right) - \frac{\partial\mathcal{L}}{\partial\phi_a} = 0. \tag{6}$$

Example (II): the Klein-Gordon equation

Consider the Lagrangian for a real scalar field, $\phi(\vec{x}, t)$,

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2 \\ &= \tfrac{1}{2}\dot{\phi}^2 - \tfrac{1}{2}(\nabla\phi)^2 - \tfrac{1}{2}m^2\phi^2\end{aligned} \tag{7}$$

Where recall that we use the Minkowski space-time metric,

$$\eta^{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{8}$$

We can compare the Lagrangian in Equation 7 to the usual expression for the Lagrangian, $L = T - V$, and identify the kinetic energy of the field as

$$T = \int d^3x \frac{1}{2}\dot{\phi}^2, \tag{9}$$

and the potential energy of the field as

$$V = \int d^3x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right). \tag{10}$$

The first term in Equation 10 is called the gradient energy, while the phrase "potential energy", or just "potential", is usually reserved for the last term. This also provides some "intuitive" explanation for the choice of sign in $\eta^{\mu\nu}$, which is opposite to that used in GR: using the sign convention in Equation 8, the kinetic term in Equation 9 is positive.

In order to determine the Equation of motion associated with the Lagrangian in Equation 7, we compute:

$$\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi \equiv (\dot{\phi}, -\nabla\phi). \tag{11}$$

The resulting Euler-Lagrange equation is thus

$$\ddot{\phi} - \nabla^2\phi + m^2\phi = 0, \tag{12}$$

which can be written in a relativistic form as

$$\partial_\mu\partial^\mu\phi + m^2\phi = 0. \tag{13}$$

Equation 13 is known as **Klein-Gordon Equation** (which was first written by Schrödinger). Note though that the field ϕ **has got nothing to do with the wave function** ψ in quantum mechanics.

The Laplacian in Minkowski space is sometimes denoted by $\Box$. In this notation, the Klein-Gordon Equation read $\Box \phi + m^2 \phi = 0$.

An obvious generalization of the Klein-Gordon equation comes from considering the Lagrangian with arbitrary potential $V(\phi)$,

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \quad \Rightarrow \quad \partial_\mu \partial^\mu \phi + \frac{\partial V}{\partial \phi} = 0. \tag{14}$$

Example (III): first order Lagrangians

We could also consider a Lagrangian that is linear in time derivatives, rather than quadratic. Take a complex scalar field ψ whose dynamics is defined by the real Lagrangian,

$$\mathcal{L} = \frac{i}{2} \left(\psi^\star \dot{\psi} - \dot{\psi}^\star \psi \right) - \nabla \psi^\star \cdot \nabla \psi - m \psi^\star \psi. \tag{15}$$

The equation of motion can be determined by treating ψ and $\psi^\star$ as independent objects, so that

$$\frac{\partial \mathcal{L}}{\partial \psi^\star} = \frac{i}{2} \dot{\psi} - m\psi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{\psi}^\star} = -\frac{i}{2} \psi \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \nabla \psi^\star} = -\nabla \psi. \tag{16}$$

This gives the equation of motion,

$$i \frac{\partial \psi}{\partial t} = -\nabla^2 \psi + m\psi. \tag{17}$$

Equation 17 looks very much like the Schrödinger equation - only it is not !. Or, at least, the interpretation of this equation is very different: the field ψ is a classical field with none of the probability interpretation of the wavefunction. We will return to this point later on.

The initial data required on a Cauchy surface [1] differs for the two examples above. When $\mathcal{L} \sim \dot{\phi}^2$, both ϕ and $\dot{\phi}$ must be specified to determine the future evolution; however when $\mathcal{L} \sim \psi^\star \psi$, only ψ and $\psi^\star$ are needed.

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[1]Intuitively, a Cauchy surface is a plane in space-time which is like an instant of time; its significance is that giving the initial conditions on this plane determines the future (and the past) uniquely.

Example (IV): Maxwell's Equations

Maxwell's equations in vacuum can be derived from the Lagrangian,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2 \tag{18}$$

Note the following: (I) the minus sign in front of the first term ensures that the kinetic terms (with time derivatives) for A_i are positive, using Minkowski space-time metric (Equation 8), and thus $\mathcal{L} \sim \frac{1}{2}\dot{A}_i^2$. (II) The Lagrangian in Equation 18 has no kinetic term $\dot{A}_0^2$ for A_0, since the two terms cancel.

We can now compute

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\partial^\mu A^\nu + (\partial_\rho A^\rho)\eta^{\mu\nu}, \tag{19}$$

from which we obtain

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = -\partial^2 A^\nu + \partial^\nu(\partial_\rho A^\rho) = -\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) \equiv -\partial_\mu F^{\mu\nu} \tag{20}$$

where the **electromagnetic field strength tensor** is defined by $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Since $\mathcal{L}$ is independent on the fields (only on their derivatives), Lagrange's equation of motion becomes $\partial_\mu F^{\mu\nu} = 0$. This is equivalent to the remaining two Maxwell's equations in vacuum, $\vec{\nabla} \cdot \vec{E} = 0$, and $\partial\vec{E}/\partial t = \vec{\nabla} \times \vec{B}$.[2] Using the notation of the field strength, we may rewrite the Maxwell Lagrangian (up to an integration by parts) in the compact form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{21}$$

1.2. Locality

In each of the examples above, the Lagrangian is local. This means that there are no terms in the Lagrangian coupling $\phi(\vec{x}, t)$ directly to $\phi(\vec{y}, t)$ with $\vec{x} \neq \vec{y}$. For example, there are no terms that look like

$$L = \int d^3x d^3y \phi(\vec{x})\phi(\vec{y}). \tag{22}$$

A priori, there is no reason for this. After all, $\vec{x}$ is merely a *label*, and we are quite happy to couple other labels together (for example, the term $\partial_3 A_0 \partial_0 A_3$ in the Maxwell Lagrangian

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[2] For further discussion, please compare with the GR notes on special relativity.

couples the $\mu = 0$ field to the $\mu = 3$ field). But the closest we get for the $\vec{x}$ label is a coupling between $\phi(\vec{x})$ and $\phi(\vec{x} + \delta\vec{x})$ through the gradient term $(\nabla\phi)^2$. *This property of locality is, as far as we know, a key feature of all theories of Nature.* Indeed, one of the main reasons for introducing field theories in classical physics is to implement locality. In this course, we will only consider local Lagrangians.

2. Lorentz invariance

The laws of nature are relativistic. In fact, **one of the main motivations to develop quantum field theory is to reconcile quantum mechanics with special relativity. To this end, we want to construct field theories in which space and time are placed on an equal footing** and the theory is invariant under Lorentz transformations,

$$x^\mu \to x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu, \tag{23}$$

where the requirement that the **interval** is conserved implies that the elements of the matrix Λ satisfy

$$\Lambda^\mu{}_\sigma \eta^{\sigma\tau} \Lambda^\nu{}_\tau = \eta^{\mu\nu}. \tag{24}$$

The matrices which satisfy Equation 24 are known as **Lorentz transformations**. Examples of Lorentz transformations are rotation by an angle θ about the x^3 axis, and a boost by $v(<1)$ along the x^1 axis; these are respectively described by the matrices

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{25}$$

where $\gamma \equiv \cosh\phi \equiv 1/\sqrt{1 - v^2}$. The set of Lorentz transformations form a group under matrix multiplication, known as **Lorentz group**. By itself the Lorentz transformations form a **Lie group** under matrix multiplications.

You have seen in previous courses how the Lorentz transformation act *on the space.* However, here we are interested in seeing how the Lorentz transformation **act on the fields that live in the space**. Mathematically, we need to find a *representation* of the Lorentz group on the fields.

The simplest example is the *scalar field* which, under the Lorentz transformation $x \to x' = \Lambda x$, transforms as $\phi'(x' = \Lambda x) = \phi(x)$, or

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x). \tag{26}$$

The inverse Λ^{-1} appears in the argument because we are dealing with an **active** transformation in which the field is truly shifted, while the coordinates are still (see Figure 1). To see why this means that the inverse appears, it will suffice to consider a non-relativistic example such as a temperature field. Suppose we start with an initial field $\phi(\vec{x}) = T(\vec{x})$ which has a hotspot at, say, $\vec{x} = (1, 0, 0)$. After a rotation, $\vec{x} \to R\vec{x}$ about the z-axis, the new field, $\phi'(\vec{x})$ will have the hotspot at, say $\vec{x} = (0, 1, 0)$.

We want now to express the new field $\phi'(\vec{x})$ in terms of the old field $\phi(\vec{x})$. We thus place ourselves at $\vec{x}' = (0, 1, 0)$ and ask what the old field looked like where we have come from at $R^{-1}(0, 1, 0) = (1, 0, 0)$. This R^{-1} is the origin of the inverse transformation. In other words, the transformed field, evaluated at the new (boosted) point x, must give the same result as the original field evaluated at the point before it was boosted $(\Lambda^{-1}x)$. (If we were instead dealing with a passive transformation in which we relabel our choice of coordinates, we would have instead $\phi(\vec{x}) \to \phi'(\vec{x}) = \phi(\Lambda x)$).
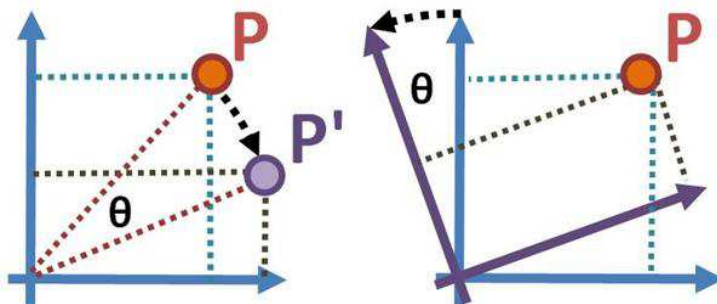

Fig. 1.— Left: In *active* transformation, a point moves from position P to P' (e.g., by rotating clockwise). In *passive* transformation (right), the point P does not move, while the coordinate system rotates counterclockwise. Note that in the active rotation, the coordinates of the point P' are the same as those of point P relative to the rotated coordinates in the passive rotation.

While the Lorentz transformation of scalar fields presented in Equation 26 is general, we are interested in particular type of theories: **Lorentz invariant theories**. The definition of **a Lorentz invariant theory** is that **if $\phi(x)$ solves the equations of motion then $\phi(\Lambda^{-1}x)$ also solves the equations of motion**. We can ensure that this property holds by **requiring that the action S is Lorentz invariant**. Let's look at our examples:

Example (I): the Klein-Gordon Equation

For a real scalar field we have $\phi(x) \to \phi(x) = \phi(\Lambda^{-1}x)$. The derivative of the scalar field is a **covariant vector** (consult with the SR notes), and therefore its transformation law is

$$(\partial_\mu \phi)(x) \to (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(y),$$

where $y = \Lambda^{-1}x$ (Compare with SR, Equations 40, 45). This means that the derivative terms in the Lagrangian density (Equation 7) transform as

$$
\begin{aligned}
\mathcal{L}_{deriv}(x) = \partial_\mu \phi(x) \partial_\nu \phi(x) \eta^{\mu\nu} \quad &\to \quad (\Lambda^{-1})^\rho{}_\mu (\partial_\rho \phi)(y)(\Lambda^{-1})^\sigma{}_\nu (\partial_\sigma \phi)(y) \eta^{\mu\nu} \\
&= \quad (\partial_\rho \phi)(y)(\partial_\sigma \phi)(y) \eta^{\rho\sigma} \\
&= \quad \mathcal{L}_{deriv}(y)
\end{aligned}
\tag{27}
$$

The potential terms $(\frac{1}{2}m^2\phi^2)$ transform according to Equation 26, with $\phi^2(x) \to \phi^2(y)$. Putting this all together, we find that the action is indeed invariant under Lorentz transformations,

$$S = \int d^4x \mathcal{L}(x) \to \int d^4x \mathcal{L}(y) = \int d^4y \mathcal{L}(y) = S, \tag{28}$$

where, in the last step, we need the fact that we don't pick up a Jacobian factor when we change integration variables from $\int d^4x$ to $\int d^4y$. This follows because $\det \Lambda = 1$. (At least for Lorentz transformation connected to the identity which, for now, is all we deal with).

Example (II): first order Lagrangians

In the first-order Lagrangian (Equation 15), space and time are *not* on the same footing. ($\mathcal{L}$ is linear in time derivatives, but quadratic in spatial derivatives). The theory is thus not Lorentz invariant.

In practice, it is easy to see if the action is Lorentz invariant: just make sure all the Lorentz indices $\mu = 0, 1, 2, 3$ are contracted with Lorentz invariant objects, such as the metric $\eta_{\mu\nu}$. Other Lorentz invariant objects you can use include the totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$ and the matrices γ_μ that we will introduce when we come to discuss spinors later on.

Example (III): Maxwell's Equations

Since the field A^μ is a contra-variant 4-vector, under a Lorentz transformation, $A^\mu(x) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$ (compare to SR, Equation 24). You can check that Maxwells Lagrangian

(Equation 21) is indeed invariant. Of course, historically electrodynamics was the first Lorentz invariant theory to be discovered: it was found even before the concept of Lorentz invariance.

3. Symmetries and Noether's Theorem

The role of symmetries in field theory is possibly even more important than in particle mechanics. There are Lorentz symmetries, internal symmetries, gauge symmetries, supersymmetries.... We start here by recasting Noethers theorem in a field theoretic framework.

3.1. Noether's Theorem

The relationship between symmetries and conservation laws in classical field theory are summarized in **Noether's theorem**. Noether's theorem states that **every continuous symmetry of the Lagrangian gives rise to a conserved current** $j^\mu(x)$ such that the equations of motion imply

$$\boxed{\partial_\mu j^\mu = 0,} \tag{29}$$

or, in other words, $\partial j^0/\partial t + \vec{\nabla} \cdot \vec{j} = 0$.

Note. A conserved current implies a conserved **charge** Q, defined as

$$Q = \int_{R^3} d^3x\, j^0. \tag{30}$$

This can be immediately seen by taking the time derivative,

$$\frac{dQ}{dt} = \int_{R^3} d^3x \frac{\partial j^0}{\partial t} = -\int_{R^3} d^3x \vec{\nabla} \cdot \vec{j} = 0, \tag{31}$$

assuming that $\vec{j} \to 0$ sufficiently quickly as $|\vec{x}| \to \infty$.

Note, though, that the existence of a current is a much *stronger statement* than the existence of a conserved charge because it implies that charge is conserved **locally**. To see this, we can define the charge in a finite volume V,

$$Q_V = \int_V d^3x\, j^0. \tag{32}$$

Repeating the analysis above, we find that

$$\frac{dQ_V}{dt} = -\int_V d^3x \vec{\nabla} \cdot \vec{j} = -\int_A \vec{j} \cdot d\vec{S}, \tag{33}$$

where A is the area bounding V, and we used Stokes' theorem. Equation 33 implies that any charge leaving V must be accounted for by a flow of the current 3-vector $\vec{j}$ out of the volume. This kind of local conservation of charge holds in any local field theory.

Proof of Noether's Theorem.
We consider **infinitesimal**[3] transformations (e.g., rotation) of the form

$$\phi_a(x) \to \phi_a'(x) = \phi_a(x) + \epsilon \delta \phi_a(x) \tag{34}$$

where ϵ is an infinitesimal parameter and $\delta \phi_a$ is some deformation of the field configuration. We call this transformation **a symmetry**, if it leaves the equations of motion invariant: $\delta \mathcal{L} = 0$. This is insured if <u>the action is invariant</u> under Equation 34.

In fact, we can allow the action to change by a surface term, as such a term will not affect the derivation of the Euler-Lagrange equations of motion. Thus, the Lagrangian must be invariant under Equation 34, up to a 4-divergence:

$$\mathcal{L}(x) \to \mathcal{L}(x) + \epsilon \partial_\mu \mathcal{J}^\mu(x), \tag{35}$$

for some set of functions $\mathcal{J}^\mu(\phi_a(x))$.

Consider now making an *arbitrary transformation*, $\phi_a(x) \to \phi_a(x) + \epsilon \delta \phi_a(x)$. Let us calculate the change in the Lagrangian:

$$
\begin{aligned}
\epsilon \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi_a}(\epsilon \delta \phi_a) + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \partial_\mu(\epsilon \delta \phi_a) \\
&= \epsilon \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right] \delta \phi_a + \epsilon \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a \right)
\end{aligned}
\tag{36}
$$

The first term (in the square brackets) vanishes by the Euler-Lagrange Equations (6). We thus have

$$\delta \mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a \right) \tag{37}$$

But by the definition of symmetry transformation, Equation 35 implies that $\delta \mathcal{L}$ must be equal to $\partial_\mu \mathcal{J}^\mu$, and we thus get that

$$\partial_\mu j^\mu = 0 \quad \text{for} \quad j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \delta \phi_a - \mathcal{J}^\mu. \tag{38}$$

Equation 38 thus implies that the current j^μ is conserved; moreover, it also tells us what it is !.

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[3]That's where we make use of the "continuous" property in Noether's theorem. E.g., when you look in the mirror, there is a *parity* symmetry, but it does not give rise to a conserved charge.

3.2. Example: Translations and the Energy-Momentum Tensor

Let us demonstrate Noether's theorem using the following example. Recall that in classical particle mechanics, invariance under **spatial translations** gives rise to the **conservation of momentum**, while invariance under **time translations** is responsible for the conservation of **energy**. We will now see the analogue in field theories.

We can describe an infinitesimal translation, $x^\nu \to x^\nu - \epsilon^\nu$, alternatively as a transformation of the field,

$$\phi_a(x) \to \phi_a(x) + \epsilon^\nu \partial_\nu \phi_a(x). \tag{39}$$

(Note that the sign in the field transformation is plus, instead of minus, because we are doing an active, as opposed to passive, transformation). Similarly, the Lagrangian - being a scalar (and assuming it has no explicit x dependence) - must transform in the same way,

$$\mathcal{L}(x) \to \mathcal{L}(x) + \epsilon^\nu \partial_\nu \mathcal{L}(x) = \mathcal{L}(x) + \epsilon^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L}(x)). \tag{40}$$

(This is the correct transformation for a Lagrangian that has *no explicit x dependence*, but only depends on x through the fields $\phi_a(x)$. All theories that we consider in this course will have this property).

Since the change in the Lagrangian is a total derivative, we may invoke Noether's theorem. In this case, we have 4 independent translations - ϵ^ν with $\nu = 0, 1, 2, 3$, (three spatial translations and one translation in time), resulting in **4 conserved currents**: $(j^\mu)_\nu$, one for each of the translations. Using $\mathcal{J}^\mu = \delta^\mu_\nu \mathcal{L}(x)$ from Equation 40, we can write the 4 currents as

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\nu \phi_a - \delta^\mu_\nu \mathcal{L} \equiv T^\mu{}_\nu. \tag{41}$$

$T^\mu{}_\nu$ is called the **energy momentum tensor** (or stress-energy tensor) of the field ϕ_a. It satisfied

$$\partial_\mu T^\mu{}_\nu = 0. \tag{42}$$

The four conserved quantities are given by

$$E = \int d^3x \, T^{00} \quad \text{and} \quad P^i = \int d^3x \, T^{0i}, \tag{43}$$

where E is the total energy of the field configuration, and P^i is the total momentum of the field configuration.

3.2.1. An Example of the Energy-Momentum Tensor

Consider the simplest scalar field theory with Lagrangian given by Equation 7, $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$. From the above discussion and Equation 41 we find

$$T^{\mu\nu} = \partial^\mu\phi\partial^\nu\phi - \eta^{\mu\nu}\mathcal{L}. \tag{44}$$

One can verify using the equation of motion for ϕ that this expression indeed satisfies $\partial_\mu T^{\mu\nu} = 0$. For this example, the conserved energy and momentum are given by

$$E = \int d^3x \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right), \tag{45}$$

$$P^i = \int d^3x\dot{\phi}\partial^i\phi \tag{46}$$

Note that E and P^i are *global* quantities, obtained by integrating over the entire space.

Further note that for this field, $T^{\mu\nu}$ is symmetric, namely $T^{\mu\nu} = T^{\nu\mu}$. This is not always the case. Nevertheless, there is typically a way to massage the energy momentum tensor of any theory into a symmetric form by adding an extra term

$$\Theta^{\mu\nu} = T^{\mu\nu} + \partial_\rho\Gamma^{\rho\mu\nu}, \tag{47}$$

where $\Gamma^{\rho\mu\nu}$ is some function of the fields that is anti-symmetric in the first two indices so $\Gamma^{\rho\mu\nu} = -\Gamma^{\mu\rho\nu}$. This guarantees that $\partial_\mu\partial_\rho\Gamma^{\rho\mu\nu} = 0$, so that the new energy-momentum tensor is also a conserved current.

A Trick.[4]
One reason we are interested in a symmetric energy-momentum tensor is to make contact with general relativity: such an object sits on the right-hand side of Einsteins field equations. In fact this observation provides a quick and easy way to determine a symmetric energy-momentum tensor. First, we transform from flat Minkowski spacetime to an arbitrary curved spacetime in the usual way: we replace $\eta_{\mu\nu}$ with the curved metric $g_{\mu\nu}(x)$, and replace the kinetic terms with suitable covariant derivatives. Then a symmetric energy momentum tensor in the flat space theory is given by

$$\Theta^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{\mu\nu}}\bigg|_{g_{\mu\nu}=\eta_{\mu\nu}}. \tag{48}$$

It should be noted however that this trick requires a little more care when working with spinors (to be introduced below).

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[4]If you study GR, this is clear. If not - sorry... you may skip this part for now.

3.3. Example II: Lorentz Transformations and Angular Momentum

In classical particle mechanics, **rotational invariance** gave rise to conservation of **angular momentum**. What is the analogy in field theory? Moreover, we now have further Lorentz transformations, namely **boosts**. What conserved quantity do they correspond to?

To answer these questions, we first need the infinitesimal form of the Lorentz transformations:

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu, \tag{49}$$

where $\omega^\mu{}_\nu$ is infinitesimal. Since Λ is a Lorentz transformation, it must satisfy Equation 24, and thus

$$(\delta^\mu{}_\sigma + \omega^\mu{}_\sigma)(\delta^\nu{}_\tau + \omega^\nu{}_\tau)\eta^{\sigma\tau} = \eta^{\mu\nu}$$
$$\Rightarrow \quad \omega^{\mu\nu} + \omega^{\nu\mu} = 0. \tag{50}$$

Thus, the infinitesimal change $\omega^{\mu\nu}$ of Lorentz transformation must be anti-symmetric matrix. As a quick check, the number of different 4×4 anti-symmetric matrices is $4 \times 3/2 = 6$, which agrees with the number of different Lorentz transformations (3 rotations + 3 boosts).

Now the transformation on a scalar field is given by

$$\begin{aligned} \phi(x) \to \phi'(x) &= \phi(\Lambda^{-1}x) \\ &= \phi(x^\mu - \omega^\mu{}_\nu x^\nu) \\ &\simeq \phi(x^\mu) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x), \end{aligned} \tag{51}$$

from which we find

$$\delta\phi = -\omega^\mu{}_\nu x^\nu \partial_\mu \phi. \tag{52}$$

We can now use the same argument for the transformation of the Lagrangian density (which is a scalar), to get

$$\delta\mathcal{L} = -\omega^\mu{}_\nu x^\nu \partial_\mu \mathcal{L} = -\partial_\mu(\omega^\mu{}_\nu x^\nu \mathcal{L}) \tag{53}$$

where the last equality follows because $\omega^\mu{}_\mu = 0$ due to anti-symmetry. Once again, the Lagrangian changes by a total derivative so we may apply Noether's theorem (now with $\mathcal{J}^\mu = -\omega^\mu{}_\nu x^\nu \mathcal{L}$), to find the conserved current,

$$\begin{aligned} j^\mu &= -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\omega^\rho{}_\nu x^\nu \partial_\rho \phi + \omega^\mu{}_\nu x^\nu \mathcal{L} \\ &= -\omega^\rho{}_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} x^\nu \partial_\rho \phi - \delta^\mu_\rho x^\nu \mathcal{L} \right] = -\omega^\rho{}_\nu T^\mu{}_\rho x^\nu. \end{aligned} \tag{54}$$

Note that as opposed to the translation case (Equation 41), we have left the infinitesimal choice of $\omega^\rho{}_\nu$ in the expression for the current in Equation 54. But clearly, we don't need it there (the same way we didn't need ϵ^ν in Equation 41). What we do need is 6 different

currents - one for each independent choice of $\omega^\rho{}_\nu$. Using the anti-symmetry property of $\omega^\rho{}_\nu$, these currents can be written as

$$(\tilde{j}^\mu)^{\rho\sigma} = x^\rho T^{\mu\sigma} - x^\sigma T^{\mu\rho}, \tag{55}$$

which satisfy $\partial_\mu (\tilde{j}^\mu)^{\rho\sigma} = 0$ and give rise to 6 conserved charges.

For $\rho, \sigma = 1, 2, 3$, the Lorentz transformation is a rotation, and the three conserved charges (defined for $\mu = 0$, see Equation 30) give the total angular momentum of the field:

$$Q^{ij} = \int d^3x (x^i T^{0j} - x^j T^{0i}) \tag{56}$$

The other three conserved charges are for boosts:

$$Q^{0i} = \int d^3x (x^0 T^{0i} - x^i T^{00}) \tag{57}$$

The fact that these are conserved implies that

$$\begin{aligned} 0 = \frac{dQ^{0i}}{dt} &= \int d^3x \, T^{0i} + t \int d^3x \frac{\partial T^{0i}}{\partial t} - \frac{d}{dt} \int d^3x \, x^i T^{00} \\ &= P^i + t\frac{dP^i}{dt} - \frac{d}{dt} \int d^3x \, x^i T^{00}. \end{aligned} \tag{58}$$

We already know that P^i is conserved, and thus $dP^i/dt = 0$; Equation 58 thus implies that under boost,

$$\frac{d}{dt} \int d^3x \, x^i T^{00} = \text{constant} \tag{59}$$

This is the statement that the center of energy of the field travels with a constant velocity. Its kind of like a field theoretic version of Newtons first law but, rather surprisingly, appearing here as a conservation law.

3.4. Internal (Global) Symmetries

The two examples given above involved transformations of spacetime, as well as transformations of the field. An **internal symmetry** is one that **only involves a transformation of the fields** and acts the same at every point in spacetime. The simplest example occurs for a complex scalar field $\psi(x) = (\phi_1(x) + i\phi_2(x))/\sqrt{2}$. We can build a real Lagrangian by

$$\mathcal{L} = \partial_\mu \psi^\star \partial^\mu \psi - V(|\psi|^2) \tag{60}$$

where the potential is a general polynomial in $|\psi|^2 = \psi^\star \psi$. To find the equations of motion, we could expand ψ in terms of ϕ_1 and ϕ_2 and work as before. However, it's easier (and

equivalent) to treat ψ and $\psi^\star$ as *independent variables*[5] and vary the action with respect to both of them. For example, varying with respect to $\psi^\star$ leads to the equation of motion

$$\partial_\mu \partial^\mu \psi + \frac{\partial V(\psi^\star \psi)}{\partial \psi^\star} = 0. \tag{61}$$

The Lagrangian has a *continuous symmetry* which rotates ϕ_1 and ϕ_2 or, equivalently, rotates the phase of ψ:

$$\psi \to e^{i\alpha} \psi \quad \text{or} \quad \delta\psi = i\alpha\psi; \quad \delta\psi^\star = -i\alpha\psi^\star \tag{62}$$

(where the last equation holds for infinitesimal constant α, which is *not* a function of x - this is what "global" means!). The Lagrangian remains invariant under this change: $\delta\mathcal{L} = 0$ (or: $\mathcal{J}^\mu = const$). The associated conserved current is

$$j^\mu = i(\partial^\mu \psi^\star)\psi - i\psi^\star(\partial^\mu \psi). \tag{63}$$

Conserved currents of this type are of extreme importance in QFT. We will see later that the conserved charges arising from currents of this type have the interpretation of **electric charge** or **particle number** (for example, baryon or lepton number; what we really mean is that the number of baryons minus anti-baryons is fixed).

Non Abelian Internal Symmetries[6]

Consider a theory involving N scalar fields ϕ_a, all with the same mass, and Lagrangian:

$$\mathcal{L} = \frac{1}{2} \sum_{a=1}^{N} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} \sum_{a=1}^{N} m^2 \phi_a^2 - g \left(\sum_{a=1}^{N} \phi_a^2 \right)^2 \tag{64}$$

In this case the Lagrangian is invariant under the non-Abelian symmetry group $G = SO(N)$. (Actually $O(N)$ in this case).[7] One can construct theories from complex fields in a similar manner that are invariant under an $SU(N)$ symmetry group. Non-Abelian symmetries of

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[5]This is why the Lagrangian in Equation 60 does not have a $(1/2)$ prefactor, as in the Lagrangian for the Klein-Gordon field.

[6]Abelian group is a commutative group, namely the result of applying an operation to two of the group members does not depend on their order.

[7]"SO" stands for "Special Orthogonal". Orthogonal group is a group of distance-preserving transformations which preserved the origin (such as rotations in Euclidean space). The group operation can be described by a $n \times n$ orthogonal matrix. The "special" implies that the determinant of the orthogonal matrix is $+1$ (it could also be -1).

this type are often referred to as **global** symmetries to distinguish them from the "local gauge" symmetries that we will meet later. Isospin is an example of such a symmetry, albeit realized only approximately in Nature.

A Trick for Determining the Conserved Current Associated with Internal Symmetry.

There is a quick method to determine the conserved current associated with an internal (global) symmetry. Suppose we have an internal symmetry, $\delta\phi = \alpha\phi$ for which the Lagrangian is invariant: $\delta\mathcal{L} = 0$. Here, α is a constant real number. (We may generalize the discussion easily to a non-Abelian internal symmetry for which α becomes a matrix).

The trick: re-do the transformation, assuming that α depends on spacetime: $\alpha = \alpha(x)$. The action is no longer invariant, and $\delta\mathcal{L}$ would not be equal to 0; however, the change in $\delta\mathcal{L}$ must be such that in the limit where α is x-independent, $\delta\mathcal{L} = 0$. Thus, the change must be of the form

$$\delta\mathcal{L} = (\partial_\mu\alpha)h^\mu(\phi), \tag{65}$$

for some function $h^\mu(\phi)$, since we know that $\delta\mathcal{L} = 0$ when α is constant. The change in the action is therefore

$$\delta S = \int d^4x\,\delta\mathcal{L} = -\int d^4x\,\alpha(x)\partial_\mu h^\mu \tag{66}$$

Here comes the point: we know that when the equations of motion are satisfied, $\delta S = 0$ for **all** variations - including the one we consider here, $\delta\phi = \alpha(x)\phi$. This can only happen if

$$\partial_\mu h^\mu = 0 \tag{67}$$

We see that we can identify the function $h^\mu = j^\mu$ **as the conserved current**. This way of viewing things emphasizes that it is the derivative terms, not the potential terms in the action that contribute to the current. (The potential terms are invariant even when $\alpha = \alpha(x)$). Exercise: using this trick, find the conserved current for the Lagrangian we derived above (Equation 60).

4. The Hamiltonian Formalism

The link between the Lagrangian formalism and the quantum theory goes via the *path integral*. In this course we will not discuss path integral methods, and focus instead on **canonical quantization**. For this we need the Hamiltonian formalism of field theory. We start by defining the *momentum* $\pi^a(x)$ conjugate to $\phi_a(x)$,

$$\pi^a(x) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}_a}. \tag{68}$$

The conjugate momentum $\pi^a(x)$ is a function of x, just like the field $\phi_a(x)$ itself. It is not to be confused with the total momentum P^i defined in Equation 43, which is a single number characterizing the whole field configuration. The **Hamiltonian density** is given by

$$\mathcal{H} = \pi^a(x)\dot{\phi}_a(x) - \mathcal{L}(x), \tag{69}$$

where, as in classical mechanics, we view $\mathcal{H}$ as a function of $\phi_a(x)$ and $\pi^a(x)$, rather than $\phi_a(x)$ and $\dot{\phi}_a(x)$. The Hamiltonian is then simply

$$H = \int d^3x \, \mathcal{H} \tag{70}$$

Example: A Real Scalar Field.

Consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - V(\phi). \tag{71}$$

The conjugate momentum is given by $\pi = \dot{\phi}$, which gives the Hamiltonian,

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi)\right]. \tag{72}$$

Clearly, the Hamiltonian agrees with the definition of the total energy (Equation 45) that we get from applying Noether's theorem for the time translation invariance.

In the Lagrangian formalism, Lorentz invariance is clear for all to see since the action is invariant under Lorentz transformations. In contrast, the Hamiltonian formalism is **not** manifestly Lorentz invariant: we have picked a preferred time. For example, the equations of motion for $\phi(x) = \phi(\vec{x}, t)$ arise from Hamilton's equations,

$$\dot{\phi}(\vec{x}, t) = \frac{\partial H}{\partial\pi(\vec{x}, t)} \quad \text{and} \quad \dot{\pi}(\vec{x}, t) = -\frac{\partial H}{\partial\phi(\vec{x}, t)}. \tag{73}$$

Unlike the Euler-Lagrange Equations (Equation 6), Equations 73 do not look Lorentz invariant. However, clearly, even though the Hamiltonian framework doesn't *look* Lorentz invariant, the physics must remain unchanged. If we start from a relativistic theory, all final answers must be Lorentz invariant even if it's not manifest at intermediate steps. We will pause at several points along the quantum route to check that this is indeed the case.