Collisions in plasmas

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November 7, 2019

This part of the course is based on Ref. [1].

1. Introduction

Let us consider what happens to the momentum and energy of a test particle of charge q and mass m, when injected with velocity v_0 into a plasma. The test particle will undergo a series of of random collisions with the plasma particles. These collisions will alter both the momentum and the energy of the test particle.

Scattering were studied in details by **Ernst Rutherford** (see Figure 1). I will not repeat the full derivative here, but only provide an order of magnitude estimates to the change in energy and momentum, based on physical arguments.

We define the **impact parameter**, b to be the perpendicular distance between the the path of the particle and the the center of the potential field (see Figure 1).

It is useful to separate the scattering events (collisions) into two categories: (1) large angle collisions, where the scattering angle is $\pi/2 \leq \theta \leq \pi$, and small angle collisions, where $\theta \ll \pi/2$.

1.1. Strong collisions

As the test particle of charge q approaches a particle of charge Q in the plasma, it is subject to Coulomb interaction, via the Coulomb potential of the particle,

$$U_E = k_e \frac{qQ}{r} \tag{1}$$

Here, $k_e = \frac{1}{4\pi\epsilon_0}$ is Coulomb's constant.

Heuristically, when the impact parameter is small enough, the scattering angle will be large. We can approximate the impact parameter that leads to strong scattering by equating

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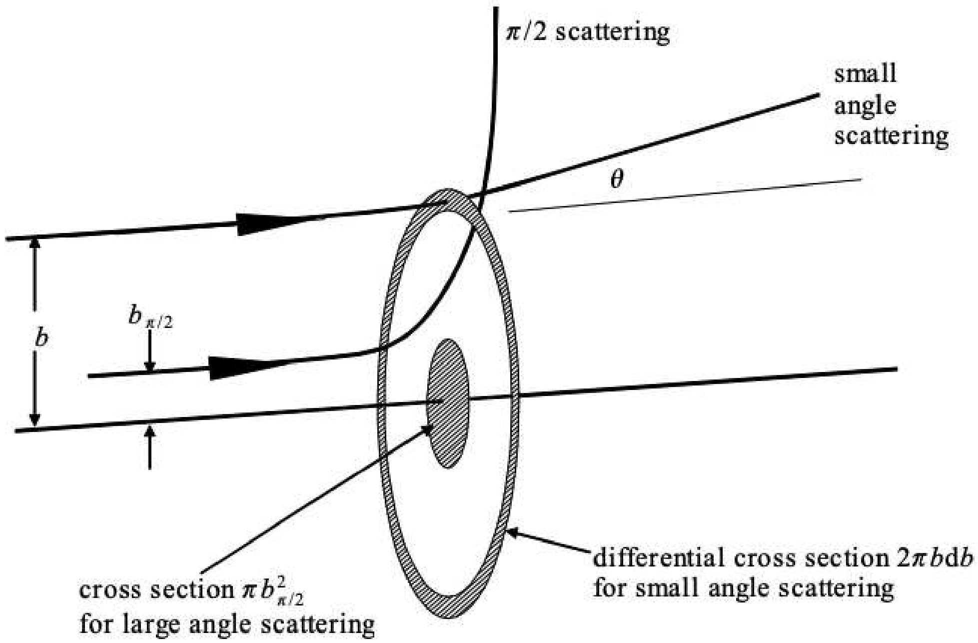

Fig. 1.— Differential scattering cross sections for large and small deflections.

the particle's kinetic energy with the potential,

$$\frac{mv_0^2}{2} = k_e \frac{qQ}{b_0} \quad \rightarrow \quad b_0 = k_e \frac{2qQ}{mv_0^2}. \tag{2}$$

For $b \gg b_0$ the scattering will be weak, while for $b \ll b_0$, the scattering will be strong.

The **cross section** for interaction is $\sigma = \pi b_0^2$. Therefore, the rate of strong collisions (number of collisions per unit time) is given by

$$\nu_s = n\sigma v_0 = \pi b_0^2 n v_0 = \pi \left(k_e \frac{2qQ}{mv_0^2} \right)^2 n v_0 \tag{3}$$

where n is the number density of particles in the plasma. In thermodynamic equilibrium, $\frac{1}{2}mv_0^2 = \frac{3}{2}k_B T \quad \rightarrow v_0 \propto T^{-1/2}$, hence the rate of strong collisions is $\nu_s \propto v_0^{-3} \propto T^{-3/2}$. We thus find that as the temperature gets higher, the rate of strong collisions decreases.

1.2. Weak collisions

We would like to compare the relative effect of weak vs. strong collisions in affecting the motion of the test particle.

The first thing to note is that small-angle scattering (weak collisions) occur **much more frequently** that strong collisions. This is because the cross section for these collisions is much larger - particles will undergo these collisions when they encounter the collision **outside** of the shaded region in Figure 1.

However, clearly, the scattering angle in each of these collisions will be very small: the particle's momentum is barely changed in each collision. In the direction parallel to the particle's momentum, the acceleration before and after the scattering nearly cancels out (for $\Delta v_\perp = 0$ is cancels out exactly). We can calculate the change in the particle's perpendicular velocity, as a function of the impact parameter (see Figure 2):

$$m\Delta v_\perp = \int F_\perp dt = \int_{-\infty}^{\infty} k_E \frac{qQ}{r^2} \sin\phi dt = \int_{-\infty}^{\infty} k_E \frac{qQ}{b^2} \sin^3\phi dt \tag{4}$$

where we have used the fact that $b = r\sin\phi$.

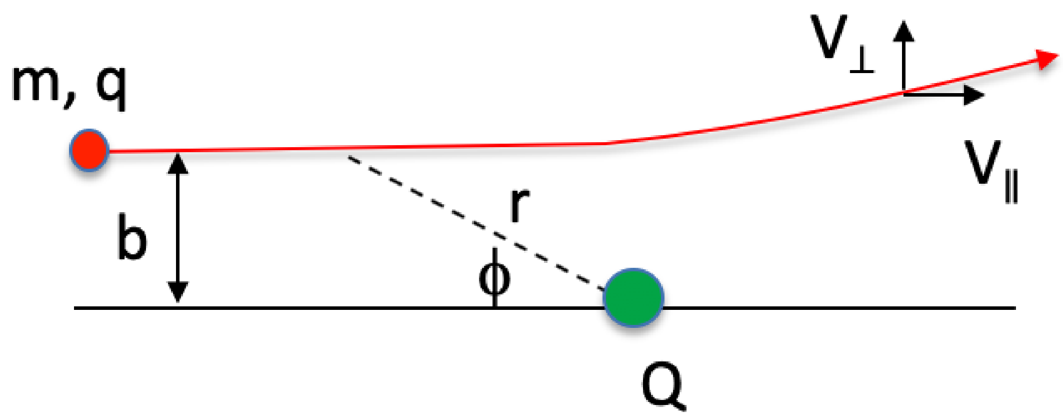

Fig. 2.— Scattering setup for a small deflection.

We thus get

$$\Delta v_\perp = k_e \frac{qQ}{mb^2} \int \sin^3\phi dt$$

Using $\tan\phi = -b/x$, for constant velocity we can write

$$x = v_0 t = -b\tan^{-1}\phi$$

from which

$$dt = \frac{b}{v_0}\frac{1}{\sin^2\phi}d\phi.$$

We thus finally obtain

$$\Delta v_\perp = k_e\frac{qQ}{mb^2}\frac{b}{v_0}\int_0^\pi \sin\phi d\phi = \frac{2k_e qQ}{mbv_0} \tag{5}$$

This is generally a small deflection, but, as stated above, there are many more weak scattering than strong ones. Because the azimuthal angle about the direction of incidence is random, the simple average of N small angle scattering vanishes. Rather, what we have is a **random walk**, where the square of the perpendicular velocity changes after N scattering in accordance to

$$\Delta(v_\perp^2) = (\Delta v_\perp)^2 N \tag{6}$$

In order to sum over the number of collisions, we recall that in each collision, the impact parameter b, hence $\Delta v_\perp$ is different. We thus need to consider the **differential cross section**, $d\sigma = 2\pi b db$ when summing over the rate of collisions. The (differential) number of scattering within time τ with impact parameter $b..b + db$ is

$$dN = d\nu\tau = \tau v_0 n d\sigma = \tau v_0 n 2\pi b db \tag{7}$$

($2\pi b db$ is the area of the ring into which the particles are scattered; see Figure 1).

We thus have

$$d[\Delta(v_\perp^2)] = (\Delta v_\perp)^2 dN = \left(\frac{2k_e qQ}{mbv_0}\right)^2 2\pi b v_0 n\tau db = \frac{8\pi n\tau k_e^2 q^2 Q^2}{m^2 v_0}\frac{db}{b} \tag{8}$$

We now need to integrate over the impact parameter. What are the physical limits ? The lower limit is $b_0 < b$, as for smaller impacts we end up with strong collisions. As an upper bound, we recall that the electric field is screened at distances larger than λ_D. We thus take λ_D as the upper bound. Carrying the integration, we get

$$\Delta(v_\perp^2) = \frac{8\pi n\tau k_e^2 q^2 Q^2}{m^2 v_0}\int_{b_0}^{\lambda_D}\frac{db}{b} = \frac{8\pi n\tau k_e^2 q^2 Q^2}{m^2 v_0}\ln\left(\frac{\lambda_D}{b_0}\right). \tag{9}$$

Finally, we can get the rate at which the weak scattering change he particle's angle significantly. This happens when $\Delta(v_\perp^2) \approx v_0^2$, namely

$$\nu_w = \frac{1}{\tau} = \frac{8\pi n k_e^2 q^2 Q^2}{m^2 v_0^3}\ln\left(\frac{\lambda_D}{b_0}\right) \tag{10}$$

We thus find a very interesting result: the rate of weak scattering is similar to that of strong scattering, up to a factor $\ln(\lambda_D/b_0)$ (and a numerical factor of the order unity). While the ratio λ_D/b_0 can be very large, the logarithm implies that it is typically of the order of few - few tens at most.

We thus conclude that under all conditions (temperature, density), the weak scattering is more important than the strong one, but only by a factor of few, due to the logarithm. This also means that in high temperature, Coulomb collisions become less and less important, and can be dominated by other phenomena.

REFERENCES

[1] P. Bellan, *Fundamentals of Plasma Physics*