

Continuum Mechanics

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1. Basic definitions

Broadly speaking materials can be divided into two categories: **solids** and **fluids**. **Solids** generally maintain their shape when subject to external forces (although they may be distorted). **Fluids** are material that can flow, and have no definite shape. They can be divided into **liquids** and **gases**.

One physical property is the **density**, defined by $\rho \equiv m/V$, where m is the mass and V is the volume of the object. The inverse quantity, ρ^{-1} is known as **specific volume** (the volume per unit mass). The density of few substances is given in Table 1.

Substance	density [kg m^{-3}]
air (0°)	1.29
air (20°)	1.21
water (4°)	10^3
aluminum (20°)	2.7×10^3
lead (20°)	1.14×10^4

Table 1: density of selective substances.

2. Longitudinal deformation in solids: the Young modulus

When an external force is applies on a solid, it may change its shape and/or volume. If the forces are not too strong, the body will return to its original shape after the distorting forces are removed. This is called **elasticity**.

Consider a piece of metal with uniform cross sectional area, A , under tension applies by equal and opposite forces, \vec{F} . If the forces change from \vec{F} to $\vec{F} + \Delta\vec{F}$, than the axial bar length changes from L to $L + \Delta L$. The fractional change in length is called **the longitudinal strain** $= \Delta L/L$. The force per unit area is called **the longitudinal stress** $= \Delta\vec{F}/A$.

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It is found experimentally that if the force is not too large, $\Delta L/L$ is linearly proportional to $\Delta\vec{F}/A$, and when the force is removed the body retracts completely to its original form. This behavior is called **elastic**. A similar behavior is found when the forces act to compress the rod.

Thus the strain is linearly proportional to the stress, and vice versa; we can write

$$\frac{|\Delta\vec{F}|}{A} = E \frac{\Delta L}{L}, \quad (1)$$

where the proportionality constant is a characteristic of the material, and is called **Young modulus**,

$$E = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{L}{A} \frac{d|\vec{F}|}{dL} \quad (2)$$

If the forces applied to a body exceed a certain strength, the body will permanently deformed. This is called **plastic** behavior. The relation between stress and strain is not linear, and the process is **irreversible**. The point in which the behavior becomes nonlinear and non-reversible is called **the elastic limit**.

Similar to the longitudinal strain, there is also a tangential strain, a volumetric and shearing deformation; all are defined in a similar way to the Young modulus.

3. Fluids at rest: pressure

Let us now discuss fluid bodies. Unlike solids, fluids cannot support longitudinal or shearing stresses - in fact, this is what *defines* their properties. Therefore, the forces on a fluid element must be perpendicular to the surface of the element; any tangential component would cause the fluid to flow (see Figure 1).

Let us denote the surface area of a face by ΔA . The fluid element is at rest due to the combined forces $\pm\Delta\vec{F}$ exerted on both sides of the fluid element by the surrounding fluid. We define the **pressure** at a point in the fluid by

$$P \equiv \lim_{\Delta A \rightarrow 0} \frac{|\Delta\vec{F}|}{\Delta A} \quad (3)$$

The pressure is a **scalar**, and has SI units of N m^{-2} which are called Pascal, and denoted Pa.

Examples.

(1) Consider a horizontal tube of liquid, with cross-sectional area ΔA . Call P_1 and P_2

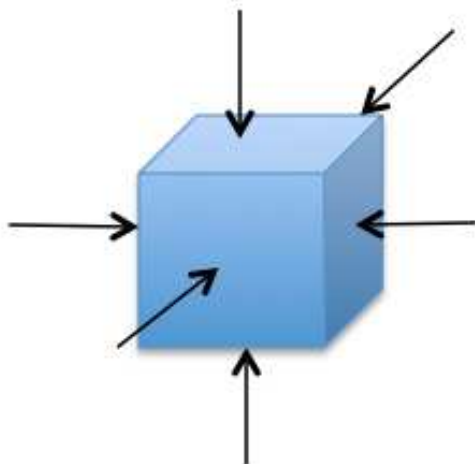


Fig. 1.— An arbitrary fluid element at rest feels forces perpendicular to its surface.

the pressures in the fluid at each side of the tube. The net horizontal force is $\Delta \vec{F} = P_2 \Delta A - P_1 \Delta A = (P_2 - P_1) \Delta A = 0$, otherwise the fluid is not at rest. Thus, $P_1 = P_2$, namely *the pressure is the same at all points in a horizontal plane in a fluid at rest.*

(2) We can conclude that *the surface of a fluid in a uniform gravitational field must be horizontal.*

(3) Consider now a fluid in uniform gravitational field. How does the pressure varies with depth ? Equating the forces on a fluid element (see Figure 2), we find $P_2 \Delta A - P_1 \Delta A - (\Delta M)g = 0$. For a *homogeneous* fluid, $\Delta M = \rho \Delta V = \rho \Delta A \Delta y$. We can thus write

$$\begin{aligned} P_2 \Delta A - P_1 \Delta A &= \rho \Delta A \Delta y g \\ \Delta P = (P_2 - P_1) &= \rho g \Delta y \end{aligned} \quad (4)$$

Thus, in a homogeneous fluid in a uniform gravitational field, the pressure increases linearly with depth.

(4) Consider a fluid that completely fills a closed container, with two freely moving pistons (see Figure 3). If one piston is pressed, it forces a pressure on the fluid. If the fluid is at rest, this pressure is transmitted undiminished to every part of the fluid. This is known as **Pascal's Principle**, after Blaise Pascal.

If the force applied by a piston is changed, the force on the other piston is adjusted appropriately if the fluid is kept at rest. Assuming the pistons are at the same height, they

apply the same pressure, namely

$$P_1 = \frac{F_1}{A_1} = \frac{F_2}{A_2} = P_2 \rightarrow \frac{F_2}{F_1} = \frac{A_2}{A_1} \quad (5)$$

Hydraulic lifts and hydraulic brakes are based on the idea presented in Equation 5: If $A_2 \gg A_1$ then $F_2 \gg F_1$, and thus applying a small force to the smaller piston results in much larger force on the larger piston, which can be used to move a heavy object. The ratio F_2/F_1 is called **mechanical advantage**.

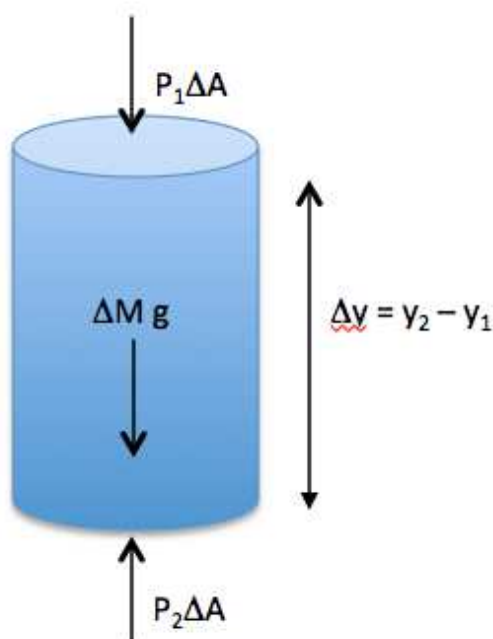


Fig. 2.— An arbitrary fluid element at rest feels forces perpendicular to its surface.

4. Mechanical work in expansion

Consider a fluid, initially having volume V that expands to volume $V + \Delta V$ against a pressure P . Take a fluid element of cross sectional area δA and thickness δz . The force exerting on the area δA is $P\delta A$, hence the work done by this force on the fluid element is $\delta W = (\delta F)(\delta z) = P(\delta A)(\delta z) = P\delta V$. The total work done is thus

$$\Delta W = P(\Delta V) \quad (6)$$

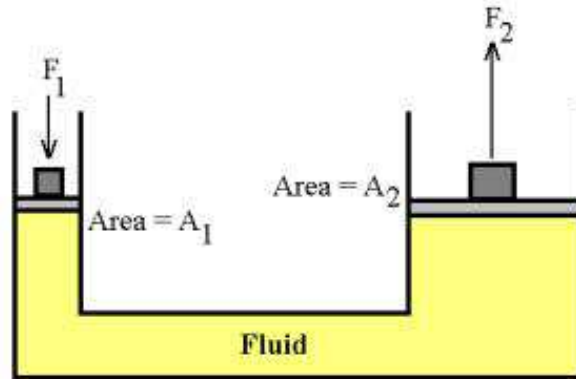


Fig. 3.— An arbitrary fluid element at rest feels forces perpendicular to its surface.

5. Atmospheric pressure

The gas in the earth's atmosphere cannot be considered static. First, there are winds; second, the density drops with height, $\rho = \rho(y)$; third, the chemical composition can vary, e.g., due to water vapor.

Near the surface, the pressure can be measured by putting one end of a transparent tube in a container filled with liquid, which is open to the atmosphere. Then, using a vacuum pump, the other side of the tube is put in vacuum. The liquid rises in the tube, until it settles into equilibrium with the pressure due to the column of the liquid balances the atmospheric pressure. Using Equation 4, the atmospheric pressure is $P = g\rho H$, where H is the height and ρ is the liquid's density. (see Figure 4).

Common units of pressure are mm Hg (mm of mercury), and bar, where $1 \text{ bar} = 10^5 \text{ Pa}$. $1 \text{ bar} \approx 760 \text{ mm Hg} \approx 1 \text{ atm}$ is the typical pressure at sea level.

6. Archimedes' principle (212 BC)

Every body (e.g., a swimmer) in the water feels a **buoyancy force** experienced when immersed in the water. This force acts to produce an *upward force*. The net upward buoyancy force is called an **upthrust**.

Consider a solid body of volume V immersed in a fluid (see Figure 5). The forces acting on the body are: (I) Its weight, $m\vec{g}$, downward; and (II) the resultant force (upthrust) on the body due to the surrounding fluid. This is a vector sum of the forces on each surface

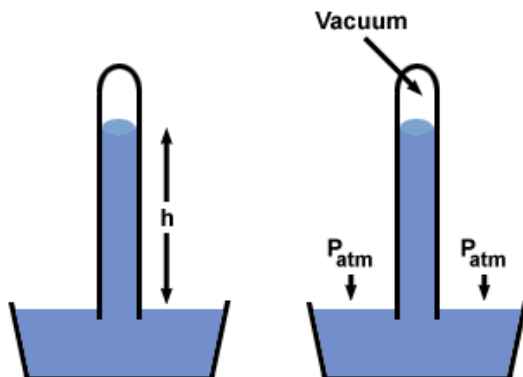


Fig. 4.— An arbitrary fluid element at rest feels forces perpendicular to its surface.

area element ΔA , directed perpendicular to the surface element: $\Delta \vec{F} = P(\Delta A)\vec{n}$.

If the shape of the body is known, one can sum up these forces; however, there is a simple way to determine the upthrust force for every shape.

If the solid body is removed from the fluid, its volume is occupied by the fluid that fills in the space. This fluid is in equilibrium (if at rest). The force that acts on the same volume V of fluid (Figure 5, right) must exactly cancel the weight of the fluid that replaced the body, of volume V . These are exactly the same forces that acted on the original solid body. Thus, **the upthrust experienced by a solid body of volume V must be equal to the magnitude of the weight of a volume V of fluid.** This is known as **the principle of Archimedes**, and can be stated as

When a body is immersed in a fluid, it experiences an upthrust which is equal to the magnitude of the weight of fluid displaced by the body.

This principle is very general, and does not depend e.g., on the assumption of uniform density. If the density can be considered constant, ρ_f (as is in most practical cases), then the net force that acts on a solid of density ρ_s immersed in a fluid is

$$\vec{F} = m\vec{g} - \rho_f V \vec{g} = \rho_s V \vec{g} - \rho_f V \vec{g}. \quad (7)$$

Note that here we took the *downward* direction as positive.

Archimedes' Principle is applicable also in cases where the solid body is only partially immersed in water (e.g., boats). In this case, the Archimedean upthrust is $\rho_f V' \vec{g}$, where V'

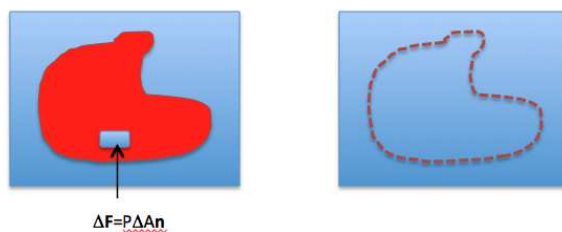


Fig. 5.— Forces acting on a body immersed in a fluid. The forces acting upon the body’s surface are equal to the forces that act on a fluid element of similar surface.

is the volume of the body below the surface. The net (upward) force is thus $\rho_f V' \vec{g} - \rho_s V \vec{g}$. If $\rho_s > \rho_f$, than the body will sink in the fluid until it reaches a position in which $\rho_f V' = \rho_s V$.

We see that the upthrust, or buoyancy force is proportional to the density of the fluid. This is why it is easier to float in salty water, since the density of salty water is greater than that of fresh water. E.g., in the Dead Sea, the density of water is $\approx 1.24 \times 10^3 \text{ kg m}^{-3}$ (as opposed to $\approx 1.0 \times 10^3 \text{ kg m}^{-3}$ in fresh water) which enables people to easily float on the water.

7. Introduction to fluid dynamics

We will consider here only slowly moving flows, namely **laminar** flow; to distinguish from **turbulent** flow, which will not be discussed in this course. In laminar flow, a fluid element proceeds along a given path, known as **streamline**.

Consider a fluid element, passing through arbitrary area A_1 at time t_1 , at velocity \vec{v}_1 (perpendicular to A_1 !). Let Δx be the distance traveled at small time interval Δt , so $\Delta x_1 = v_1 \Delta t$ (see Figure 6). At time t_2 , this same fluid element now passes through surface area A_2 , and has velocity v_2 and width $\Delta x_2 = v_2 \Delta t$. Since the **mass** of this fluid element is conserved, we can write

$$\Delta m = A_1 v_1 \Delta t \rho_1 = A_2 v_2 \Delta t \rho_2. \quad (8)$$

Taking Δt to be arbitrarily small, we find (the integral form of) **the continuity equation**,

$$\frac{dm}{dt} = A_1 v_1 \rho_1 = A_2 v_2 \rho_2. \quad (9)$$

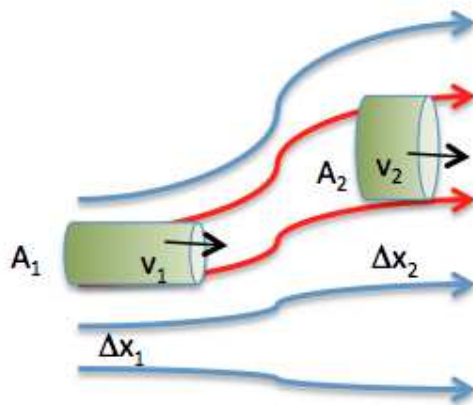


Fig. 6.— A fluid element changes its position (and shape) with time.

7.1. Bernoulli's Equation

In laminar flow, one can define **streamlines** as lines such that the tangent to a streamline at every point gives the direction of velocity at that point. Such streamlines are presented by arrows in Figure 6.

Consider a fluid element that captures the volume between the red streamlines and the surfaces A_1 and A_2 in Figure 6.

The fluid flows along the streamline, as a result of pressure exerted by the fluid to the left of A_1 , and has to overcome the pressure exerted by the fluid to the right of A_2 . According to Equation 6, the net mechanical work done on this fluid in time Δt is

$$\Delta W = P_1 A_1 (\Delta x_1) - P_2 A_2 (\Delta x_2) = P_1 A_1 v_1 \Delta t - P_2 A_2 v_2 \Delta t. \quad (10)$$

This work goes into changing the mechanical energy of the system: change in its velocity and or potential. Effectively, we can think of mass element Δm (in the green tubes in Figure 6) of being removed from the region near x_1 and added to the region near x_2 . We thus find

$$\Delta W = \left[\frac{1}{2} (\Delta m) v_2^2 + (\Delta m) g y_2 \right] - \left[\frac{1}{2} (\Delta m) v_1^2 + (\Delta m) g y_1 \right], \quad (11)$$

where y_1, y_2 are the height of the fluid element at location x_1 and x_2 .

Combining Equations 10, 11, using $\Delta m = A_1 v_1 (\Delta t) \rho_1 = A_2 v_2 (\Delta t) \rho_2$, one gets

$$\begin{aligned} \frac{P_1 A_1 v_1 \Delta t}{\Delta m} - \frac{P_2 A_2 v_2 \Delta t}{\Delta m} &= \frac{1}{2} (v_2^2 - v_1^2) + g(y_2 - y_1) \\ \frac{P_1}{\rho_1} - \frac{P_2}{\rho_2} &= \frac{1}{2} (v_2^2 - v_1^2) + g(y_2 - y_1) \\ \frac{P_1}{\rho_1} + \frac{1}{2} v_1^2 + g y_1 &= \frac{P_2}{\rho_2} + \frac{1}{2} v_2^2 + g y_2. \end{aligned} \tag{12}$$

For an **incompressible fluid**, $\rho_1 = \rho_2 = \rho$, and Equation 12 becomes

$$P + \frac{1}{2} \rho v^2 + \rho g y = \text{const.} \tag{13}$$

Equations 12 (or 13) are called **Bernoulli's equation**.

Examples.

(I) Consider an incompressible fluid at rest ($v_1 = v_2$). Equation 13 implies $P + \rho g y = \text{const}$, or $P_1 - P_2 = \rho g (y_2 - y_1)$. This is identical to Equation 4 (not surprising).

(II) In a horizontal flow, $y_1 = y_2$, and (for incompressible flow) Equation 13 becomes

$$P_1 + (1/2) \rho v_1^2 = P_2 + (1/2) \rho v_2^2. \tag{14}$$

This implies that if $v_1 > v_2$, then $P_1 < P_2$. **The faster the flow, the lower the pressure.** This result has very important applications. For example, an airplane wing is designed in such a way that air has a longer way to travel above the wing than below; hence the flow is faster above the wing, implying a lower pressure that lifts the airplane; see Figure 7.

(III) When holding two sheets of paper and blowing a wind, the two sheets will get *closer together*.

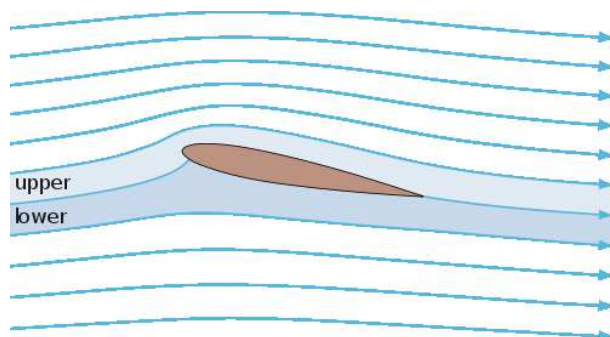


Fig. 7.— Dynamical lift on an airplane wing, The air speed above the wing is faster than below the wing, causing a pressure drop above the wing.

Flowmeters.

Using Bernoulli’s equation and mass conservation, one can construct a device to measure the flow rate in a pipe. Consider Figure 8. For an incompressible flow, one can write

$$P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2 = P_2 + \frac{1}{2}\rho \left(\frac{A_1}{A_2}v_1\right)^2, \quad (15)$$

where Equation 9 was used for incompressible fluid ($\rho_1 = \rho_2$).

The mass of fluid flowing through the pipe at any given time is thus $dM/dt = \rho A_1 v_1$, where

$$v_1 = \left(\frac{2(P_1 - P_2)}{\rho \left[\left(\frac{A_1}{A_2}\right)^2 - 1 \right]} \right)^{1/2}. \quad (16)$$

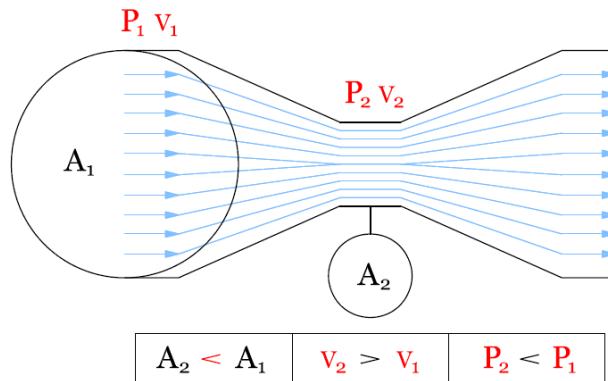


Fig. 8.— Basic idea of a flowmeter.

8. Viscosity

So far we have assumed that fluids cannot withstand a shearing force. In practice, of course, fluids never show such ideal behavior. For example, when a fluid is in a pipe, close to the wall its velocity is 0. Another example is viscous flow (e.g., honey), which flows much slower than, say, water under the same conditions.

Consider a fluid in a pipe. Very close to the wall, $\vec{v} = 0$. Slightly further away, the velocity is slightly larger, etc. Thus, there is a **velocity gradient** in the fluid (see Figure 9).

Consider a layer of fluid between y and $y + \Delta y$. (y is perpendicular to the direction of the flow). The fluid layer experiences a shearing stress $\Delta \vec{F} / \Delta A$ due to the fluid immediately above, where $\Delta \vec{F}$ is the force exerted tangentially on the area ΔA ; there is also an equal and opposite shearing stress due to the fluid immediately below.

It is found (experimentally) that provided the velocities are not too large (laminar flow), that the velocity gradient is linearly proportional to the shearing stress producing it:

$$\frac{d\vec{F}}{dA} = \eta \frac{d\vec{v}}{dy}. \quad (17)$$

The constant of proportionality, η is called the **viscosity** of the fluid. Units of viscosity are $[\text{N s m}^{-2}]$. Viscosity changes with temperature: e.g., at room temperature the viscosity of air is $\sim 2 \times 10^{-5} \text{ N s m}^{-2}$; water is $\sim 10^{-3} \text{ N s m}^{-2}$; olive oil is $\sim 8 \times 10^{-2} \text{ N s m}^{-2}$ and honey is $2 - 10 \text{ N s m}^{-2}$.

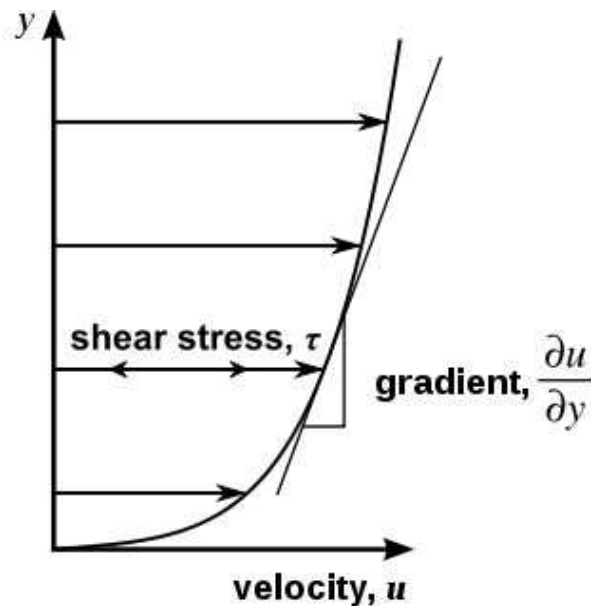


Fig. 9.— Viscosity in laminar flow

Example: A (laminar) flow in cylindrical pipe.

Consider a flow of a liquid in a cylindrical pipe of radius a and length l . The pressure difference between two ends of the pipe is assumed constant, and thus the flow is subject to a pressure gradient, dP/dl .

Consider a flow element in a small section of the pipe, of length Δl and radius r from the center of the pipe. The net force on the edges of the flow element is $(\Delta P)\pi r^2$. The

surface area is $(2\pi r)\Delta l$. Thus, the shearing stress is given by

$$\frac{dF}{dA} = \lim_{\Delta l \rightarrow 0} \frac{(\Delta P)\pi r^2}{2\pi r(\Delta l)} = \frac{r}{2} \frac{dP}{dl} = -\eta \frac{dv}{dr}, \quad (18)$$

where the minus sign $(-\eta)$ originates from the definition of r as $r = 0$ at the center of the pipe (see Figure 10). This gives

$$v = -\frac{1}{2\eta} \frac{dP}{dl} \int_a^r r dr = \frac{1}{4\eta} \frac{dP}{dl} (a^2 - r^2) \quad (19)$$

This means that the speed of the fluid takes a paraboloidal shape.

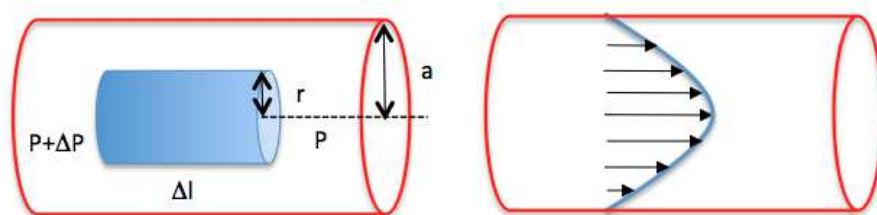


Fig. 10.— Left: A fluid element inside a pipe, Right: the velocity profile of a fluid inside a cylindrical pipe have a parabolic shape.

We can calculate the flow rate through the pipe:

$$\frac{dM}{dt} = \rho \int_0^a 2\pi r v(r) dr = \frac{2\pi\rho}{4\eta} \frac{dP}{dl} \int_0^a r(a^2 - r^2) dr = \frac{\pi\rho}{2\eta} \frac{dP}{dl} \left(\frac{a^4}{2} - \frac{a^4}{4} \right) = \frac{\pi\rho a^4}{8\eta} \frac{dP}{dl} \quad (20)$$

9. Boyle's law

We now turn the discussion into gases, rather than liquids. As discussed above, if a force acts on a piston closing a container filled with gas, it will exert pressure on the gas inside the container. This will result in a decrease in the container's volume, as the gas set into new equilibrium state.

It is found experimentally that for constant temperature, the pressure of the gas is inversely proportional to its volume, namely

$$P \propto V^{-1} \leftrightarrow PV = \text{const.} \quad (21)$$

This result is known as **Boyle’s law**.

In the limit of low pressures, *all gases* obey Boyle’s law. A deviation is seen at high pressures. A gas which obeys Boyle’s law over the entire range of pressures and volumes is called **ideal gas**.

10. Introduction to microscopic theory of gases and kinetic theory

So far we described gases by **macroscopic** quantities, such as pressure or density. However, the macroscopic properties result from collective behavior of the **microscopic** gas molecules. The **kinetic theory of gases** deals with the transition between the behavior of gas molecules on a microscopic level, and its reflection on the macroscopic world.

The basic assumptions of the theory are:

- (1) Any finite volume contains a very large number of molecules.
- (2) The size of the molecules is much smaller than the average distance between the molecules.
- (3) Collisions between the molecules and between molecules and the walls of the container are perfectly elastic.
- (4) The forces between the molecules are of very short range; basically, between each collision the molecules travel in straight lines.
- (5) Molecules are treated as rigid spheres (“billiard balls”); often, a similar mass is assumed to all molecules.

Consider a single molecule of mass m interacting with the wall. Since the interaction is assumed elastic, and the wall is at rest, we find that the momentum component parallel to the wall is unchanged, $p_{\parallel} = p'_{\parallel}$, while the momentum component perpendicular to the wall is reversed, $p_{\perp} = -p'_{\perp}$. Thus, in every collision, the wall gets momentum $\Delta p = 2p_{\perp}$.

Consider, for simplicity, a rectangular container of dimension h . If we neglect collisions between the molecules and themselves (which will be shown to be irrelevant), the time between collision of a molecule and the bottom face is $\Delta t_c = 2h/v_z$. Since in every collision the bottom face gets momentum $2p_z$, the rate of momentum transfer to the bottom face is

$$\frac{dp}{dt} = \frac{2p_z}{\Delta t_c} = \frac{p_z v_z}{h}. \quad (22)$$

The continuous transfer of momentum to the bottom face by all the molecules is equal to the **force** that the gas exert on the face. This force is thus equal to

$$F = \sum_i \frac{p_{iz} v_{iz}}{h} = \frac{N}{h} \frac{\sum_i p_{iz} v_{iz}}{N} = \frac{N}{h} \langle p_z v_z \rangle. \quad (23)$$

Here, N is the total number of molecules in the container, and $\langle p_z v_z \rangle$ is the average value of $p_z v_z$.

Since the scalar product $\vec{p} \cdot \vec{v} = p_x v_x + p_y v_y + p_z v_z$, due to symmetry (when averaging), $\langle p_z v_z \rangle = (1/3) \langle \vec{p} \cdot \vec{v} \rangle$. The force in Equation 23 is therefore

$$F = \frac{N}{3h} \langle \vec{p} \cdot \vec{v} \rangle . \quad (24)$$

The force per unit area is simply the pressure, P , and thus

$$P = \frac{F}{A} = \frac{N}{3Ah} \langle \vec{p} \cdot \vec{v} \rangle = \frac{1}{3} \frac{N}{V} \langle \vec{p} \cdot \vec{v} \rangle . \quad (25)$$

Equation 25 thus retrieves Boyle's law, from the kinetic theory of gases.

For non-relativistic molecules, $\vec{p} = m\vec{v}$, and thus Equation 25 can be written as

$$P = \frac{1}{3} \frac{N}{V} \langle mv^2 \rangle = \frac{2}{3} \frac{N}{V} \langle \frac{1}{2} mv^2 \rangle . \quad (26)$$

Since $\langle \frac{1}{2} mv^2 \rangle$ is the average kinetic energy per molecule and N is the total number of molecules, than the total mechanical energy of the gas, known as **internal energy** can be defined by

$$U \equiv N \left\langle \frac{1}{2} mv^2 \right\rangle , \quad (27)$$

and Equation 26 takes the form

$$PV = \frac{2}{3} U . \quad (28)$$

Equation 28 is valid for non-relativistic gas, namely gas in which the molecules move at non-relativistic velocities. Clearly, U is constant if there is no net energy transfer to or from the system.

If the gas molecules move at relativistic speeds, then $v \rightarrow c$ while $p \approx E/c$. In this case, $\langle \vec{p} \cdot \vec{v} \rangle \approx \langle (E/c)c \rangle = \langle E \rangle$, where $\langle E \rangle$ is the average energy per molecule. In this case, one gets $PV = (1/3)U$.

We have thus shown that Boyle's macroscopic law can be derived from a basic microscopic model.

10.1. Mole

A mole is a simple (dimensionless) number, that is defined as the number of atoms that exist in 12 gram of Carbon 12. One mole of hydrogen atoms weight one gram, etc. Easiest

to remember: $1 \text{ mole} = 1/m_p$, when $m_p = 1.67 \times 10^{-24}$ gr is the mass of a proton (measured in grams). The number of molecules in 1 mole is also called **Avogadro's constant**, denoted by N_A .

11. Van-der-Waals Equation of state

The treatment in the previous section was done for **ideal** gases. This can be improved in several ways. In particular, (1) The gas molecules apply forces on each other, and thus cannot be treated as “billiard balls”; and (2) The molecules capture a finite volume, which needs to be taken out of the available volume for free motion in the container.

Let us sketch the modification needed to Equation 28. First, due to the interactions, the effective size of the molecules, that was neglected in the former analysis, needs to be considered. This reduces the volume by a value which is proportional to the number of molecules, $V \rightarrow V' = V - \beta N$, and Equation 28 becomes

$$P = \frac{2}{3} \frac{U}{V - \beta N}. \quad (29)$$

Second, due to the attractive nature of the force, the molecules attract each other. This leads to a reduced force (hence pressure) of the molecules that interact with the walls. The net force that acts on the molecules interacting with the wall is proportional to the density of the molecules, and so does the rate of interactions of molecules with the wall.

Over all, the net force and pressure exerted on the wall is decreased by a factor $\propto (N/V)^2$, or

$$P + \alpha \left(\frac{N}{V} \right)^2 = \frac{1}{3} \frac{N}{V'} \langle \vec{p} \cdot \vec{v} \rangle = \frac{2}{3} \frac{U}{V'}, \quad (30)$$

where α_1 is a proportionality constant (compare with Equation 25).

We thus find that

$$\left(P + \alpha \frac{N^2}{V^2} \right) \left(\frac{V}{N} - \beta \right) = \frac{2}{3} \frac{U}{N}. \quad (31)$$

Equation 31 is known as *Van-der-Waals Equation of state*.