Cosmology: Part I

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This part of the course is based on Refs. [1], [2] and [3].

1. Introduction: the cosmological principle

We are now going to apply Einstein's equation to study the metric of the entire universe. The ability to study global properties such as the structure and temporal evolution of the universe on the largest scale opened a new branch of science, known as **cosmology**.

Modern day cosmology is based on the hypothesis that on a large enough scale the universe is spatially **homogeneous** and **isotropic**. Together, these two assumptions are known as **the cosmological principle**. By **homogeneity** we mean that the properties of the universe are the same at every point in space; *the universe is invariant under translations*. By **isotropy** we mean that being in a given point, in every direction we look at, the properties of the universe look the same; *the universe is invariant under rotations*.

The roots of this idea lie in the **Copernican principle**, which states that the earth is not in a central, specially favored position.

Clearly, the idea that the universe is homogeneous and isotropic seem to be incorrect: when looking to the right and to the left (or here and at the center of the sun), the conditions are obviously *not* similar, and the universe is thus *not* homogeneous and isotropic; however, we **believe it is so on the very largest scale**.

The validity of the cosmological principle on the largest scales is manifested in a number of different observations, such as (i) number counts of galaxies and (ii) observations of diffuse X-ray and γ-ray backgrounds. It is most clear in the (iii) 2.7°K microwave background radiation. Although we now know that the microwave background is not perfectly smooth (and nobody ever expected that it was), the deviations from regularity are on the order of 10^{-5} or less, certainly an adequate basis for an approximate description of spacetime on large scales.

From a formal mathematical definition, by **homogeneity** we mean that given any two points p and q in a manifold M, there is an isometry which takes p into q. Note that a metric

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can be isotropic but not homogeneous (such as a cone, which is isotropic around its vertex but certainly not homogeneous), or homogeneous but nowhere isotropic (such as $\mathbf{R} \times S^2$ in the usual metric).

On the other hand, if a space is isotropic *everywhere* then it is homogeneous. Likewise, if it is isotropic around one point and also homogeneous, it will be isotropic around every point. Since there is ample observational evidence for isotropy, and the Copernican principle would have us believe that we are not the center of the universe and therefore observers elsewhere should also observe isotropy, we will henceforth assume both homogeneity and isotropy. A space that is both homogeneous and isotropic is a *maximally symmetric* space.

There is one catch. When we look at distant galaxies, they appear to be receding from us; the universe is apparently not static, but changing with time. Therefore the idea that the universe is homogeneous and isotropic, apply only to the spatial part of space-time, which is a sub-space of the 4-dimensional space-time.

Thus, while I could introduce directly the corresponding metric (which is known as the **Robertson-Walker metric**), for mathematical completeness let us discuss first some mathematical properties of maximally symmetric spaces, and then apply them to introduce the metric.

2.   Maximally symmetric spaces

Formally, a **maximally symmetric space** is a space which possesses the largest possible number of Killing vectors. On an n-dimensional manifold this number is $n(n+1)/2$. While I will not prove this statement below, it is easy to understand at an informal level. Consider the Euclidean space $\mathbf{R}^n$, where the isometries are well known to us: translations and rotations. In general there will be n translations, one for each direction we can move. There will also be $n(n-1)/2$ rotations; for each of n dimensions there are $n-1$ directions in which we can rotate it, but we must divide by two to prevent overcounting (rotating x into y and rotating y into x are two versions of the same thing). We therefore have

$$n + \frac{n(n-1)}{2} = \frac{n(n+1)}{2} \tag{1}$$

independent Killing vectors. The same kind of counting argument applies to maximally symmetric spaces with curvature (such as spheres) or a non-Euclidean signature (such as Minkowski space), although the details are marginally different.

Note that in $n \geq 2$ dimensions, there can be more Killing vectors than dimensions. This is because a set of Killing vector fields can be linearly independent, even though at any one

point on the manifold the vectors at that point are linearly dependent. It is trivial to show (so you should do it yourself) that a linear combination of Killing vectors with *constant* coefficients is still a Killing vector (in which case the linear combination does not count as an independent Killing vector), but this is not necessarily true with coefficients which vary over the manifold.

2.1. A few more algebraic properties of Killing vectors

Let us return in this sub-section to discuss general properties of Killing vectors, to be used later in the study of maximally symmetric spaces.

Recall that Killing vectors satisfy Killing conditions,

$$K_{\sigma;\rho} + K_{\rho;\sigma} = 0. \tag{2}$$

From the definition of the curvature tensor, we have

$$K_{\sigma;\rho;\mu} - K_{\sigma;\mu;\rho} = -R^{\lambda}{}_{\sigma\rho\mu}K_{\lambda}. \tag{3}$$

(Recall that Equation 3 is true by the definition / construction of the curvature tensor for any general vector V_{σ}, and thus it is clearly true for $V_{\sigma} = K_{\sigma}$; see the chapter on "curvature", Equation 12). Using the cyclic sum rule of the curvature tensor,

$$R^{\lambda}{}_{\sigma\rho\mu} + R^{\lambda}{}_{\mu\sigma\rho} + R^{\lambda}{}_{\rho\mu\sigma} = 0, \tag{4}$$

in Equation 3, we find that any vector (not necessarily a Killing vector !) K_{μ} satisfies

$$K_{\sigma;\rho;\mu} - K_{\sigma;\mu;\rho} + K_{\mu;\sigma;\rho} - K_{\mu;\rho;\sigma} + K_{\rho;\mu;\sigma} - K_{\rho;\sigma;\mu} = 0. \tag{5}$$

Using now the Killing condition (Equation 2) in Equation 5, we get

$$K_{\sigma;\rho;\mu} - K_{\sigma;\mu;\rho} - K_{\mu;\rho;\sigma} = 0, \tag{6}$$

from which the commutator of the two covariant derivatives in Equation 3 becomes

$$K_{\mu;\rho;\sigma} = -R^{\lambda}{}_{\sigma\rho\mu}K_{\lambda}. \tag{7}$$

Using again the definition of the curvature tensor, we can write the commutator of the two covariant derivatives (again, from the definition of the curvature tensor, Equation 11 there):

$$K_{\rho;\mu;\sigma;\nu} - K_{\rho;\mu;\nu;\sigma} = -R^{\lambda}{}_{\rho\sigma\nu}K_{\lambda;\mu} - R^{\lambda}{}_{\mu\sigma\nu}K_{\rho;\lambda}. \tag{8}$$

In order for Equation 7 to fulfill the condition set in Equation 8, it is required that

$$R^\lambda{}_{\sigma\rho\mu}K_{\lambda;\nu} - R^\lambda{}_{\nu\rho\mu}K_{\lambda;\sigma} + \left(R^\lambda{}_{\sigma\rho\mu;\nu} - R^\lambda{}_{\nu\rho\mu;\sigma}\right)K_\lambda = -R^\lambda{}_{\rho\sigma\nu}K_{\lambda;\mu} - R^\lambda{}_{\mu\sigma\nu}K_{\rho;\lambda} \qquad (9)$$

(we basically took the derivative of Equation 7 and plugged it in, while replacing $\mu \leftrightarrow \rho$ and $\sigma \leftrightarrow \nu$). When using again the Killing condition (Equation 2, $K_{\sigma;\rho} + K_{\rho;\sigma}$) we find that

$$\left(-R^\lambda{}_{\rho\sigma\nu}\delta^\kappa_\mu + R^\lambda{}_{\mu\sigma\nu}\delta^\kappa_\rho - R^\lambda{}_{\sigma\rho\mu}\delta^\kappa_\nu + R^\lambda{}_{\nu\rho\mu}\delta^\kappa_\sigma\right)K_{\lambda;\kappa} = \left(R^\lambda{}_{\sigma\rho\mu;\nu} - R^\lambda{}_{\nu\rho\mu;\sigma}\right)K_\lambda \qquad (10)$$

Armed with this, let us look at maximally symmetric spaces.

2.2. Killing vectors in maximally symmetric spaces

A maximally symmetric space is both homogeneous and isotropic.

Consider a point X in a **homogeneous** space. The homogeneity of space implies that the metric must admit Killing vectors that take all possible values of X, namely at a given point in space there is a Killing vector in the direction of any other point. In n-dimensional space, there are thus n such independent Killing vectors.

The space is also **isotropic** at X; Thus, the infinitesimal isometries (coordinate change $x^{\mu'} = x^\mu + \epsilon K^\mu$) leave the point X fixed, namely $K^\lambda(X) = 0$, and at the same time the first derivative $K_{\lambda;\nu}(X)$ takes any value - subject only to the antisymmetry condition set by the Killing equation (equation 2) [Think of rotation around a given point. The Killing vector associated with it will be zero at the origin]. Combined with the homogeneity condition, this implies that $K_{\lambda;\nu} = K_{\lambda;\nu}(X)$ is an arbitrary antisymmetric matrix, and therefore in n dimensions there are $n(n-1)/2$ independent such Killing vectors.

We can now use the conditions that in a maximally symmetric space $K_\lambda = 0$ and $K_{\lambda;\nu} = -K_{\nu;\lambda}$ in Equation 10, to find that the coefficient of $K_{\lambda;\kappa}$ must have a vanishing antisymmetric part, namely

$$\left(-R^\lambda{}_{\rho\sigma\nu}\delta^\kappa_\mu + R^\lambda{}_{\mu\sigma\nu}\delta^\kappa_\rho - R^\lambda{}_{\sigma\rho\mu}\delta^\kappa_\nu + R^\lambda{}_{\nu\rho\mu}\delta^\kappa_\sigma\right) = \left(-R^\kappa{}_{\rho\sigma\nu}\delta^\lambda_\mu + R^\kappa{}_{\mu\sigma\nu}\delta^\lambda_\rho - R^\kappa{}_{\sigma\rho\mu}\delta^\lambda_\nu + R^\kappa{}_{\nu\rho\mu}\delta^\lambda_\sigma\right) \quad (11)$$

We next contract κ and μ to get

$$-nR^\lambda{}_{\rho\sigma\nu} + R^\lambda{}_{\rho\sigma\nu} - R^\lambda{}_{\sigma\rho\nu} + R^\lambda{}_{\nu\rho\sigma} = -R^\lambda{}_{\rho\sigma\nu} + R_{\sigma\rho}\delta^\lambda_\nu - R_{\nu\rho}\delta^\lambda_\sigma \qquad (12)$$

where we have used the facts that $R^\mu{}_{\mu\sigma\nu} = 0$, $-R^\kappa{}_{\sigma\rho\kappa} = R_{\sigma\rho}$, and in n dimensions $\delta^\kappa_\kappa = n$.

We use now the cyclic sum rule (Equation 4) so that the last 3 terms in the left hand side are equal to 0. Further multiply by $g_{\lambda\alpha}$, Equation 12 becomes

$$(n-1)R_{\alpha\rho\sigma\nu} = R_{\nu\rho}g_{\alpha\sigma} - R_{\sigma\rho}g_{\alpha\nu} \qquad (13)$$

(I switched the order of the last two terms in Equation 12 - the last term on the right hand side pf Equation 12 becomes the first term on the right hand side of Equation 13, and the one before last became the last).

Due to the symmetry properties of the metric tensor, this must be symmetric with respect to exchange of α and ρ, and so

$$R_{\nu\rho}g_{\alpha\sigma} - R_{\sigma\rho}g_{\alpha\nu} = R_{\nu\alpha}g_{\rho\sigma} + R_{\sigma\alpha}g_{\rho\nu}. \tag{14}$$

Multiplying by $g^{\alpha\lambda}$ and contracting λ with ν, we find

$$R_{\sigma\rho} - nR_{\sigma\rho} = -R^{\lambda}{}_{\lambda}g_{\sigma\rho} + R_{\rho\sigma}. \tag{15}$$

The Ricci tensor thus takes the form

$$R_{\sigma\rho} = \frac{1}{n}g_{\sigma\rho}R^{\lambda}{}_{\lambda} \tag{16}$$

and using Equation 13 we can write the Riemann tensor as

$$R_{\lambda\rho\sigma\nu} = \frac{R^{\alpha}{}_{\alpha}}{n(n-1)}\left(g_{\nu\rho}g_{\lambda\sigma} - g_{\sigma\rho}g_{\lambda\nu}\right). \tag{17}$$

As a final step, we note that in a space that is isotropic *everywhere*, Equation 16 holds at every point. In this case, we can use the Bianchi identity (Equation 36 in "curvature", $\nabla^{\mu}G_{\mu\nu} = \nabla^{\mu}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = 0$) with Equation 16 to write

$$0 = [R^{\sigma}{}_{\rho} - \frac{1}{2}\delta^{\sigma}_{\rho}R^{\lambda}{}_{\lambda}]_{;\sigma} = \left(\frac{1}{n} - \frac{1}{2}\right)(R^{\lambda}{}_{\lambda})_{;\sigma} \tag{18}$$

This result implies that if $n > 2$ (namely, in 3 or more dimensions) that $R^{\lambda}{}_{\lambda}$ is *constant* in space. It is thus convenient to introduce the curvature constant k by $R^{\lambda}{}_{\lambda} \equiv n(n-1)k$, and write equation 17 as

$$R_{\lambda\rho\sigma\nu} = k\left(g_{\nu\rho}g_{\lambda\sigma} - g_{\sigma\rho}g_{\lambda\nu}\right) \tag{19}$$

3. The Robertson-Walker metric

Armed with Equation 19, we can return now to the construction of a cosmological model. Our model will consider the idea that the universe is homogeneous and isotropic in space, but not in time.

We therefore consider our spacetime to be $\mathbf{R} \times \Sigma$, where $\mathbf{R}$ represents the time direction and Σ is a homogeneous and isotropic three-manifold. The usefulness of homogeneity and

isotropy is that they imply that Σ must be a maximally symmetric space. This is a direct consequence of the homogeneity and isotropy assumption. The metric thus takes the form

$$ds^2 = -dt^2 + a^2(t)g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j \ . \tag{20}$$

Here t is the timelike coordinate, and (x^1, x^2, x^3) are the coordinates on Σ; g_{ij} is the maximally symmetric metric on Σ.

The function $a(t)$ is known as the **scale factor**, and it tells us "how big" the spacelike slice Σ is at the moment t. The coordinates used here, in which the metric is free of cross terms $\mathrm{d}t\,\mathrm{d}x^i$ and the spacelike components are proportional to a single function of t, are known as **comoving coordinates**, and an observer who stays at constant x^i is also called "comoving". Only a comoving observer will think that the universe looks isotropic; in fact on Earth we are not quite comoving, and as a result we see a dipole anisotropy in the cosmic microwave background as a result of the conventional Doppler effect.

We can now write Equation 19 as

$$^{(3)}R_{ijkl} = k(g_{ik}g_{jl} - g_{il}g_{jk}) \ , \tag{21}$$

where we put a superscript $^{(3)}$ on the Riemann tensor to remind us that it is associated with the three-metric g_{ij}, not the metric of the entire spacetime. The Ricci tensor is then

$$^{(3)}R_{jl} = (n - 1)kg_{jl} = 2kg_{jl} \ . \tag{22}$$

(where $n = 3$ dimensions).

If the space is to be maximally symmetric, then it will certainly be spherically symmetric. We already know something about spherically symmetric spaces from our exploration of the Schwarzschild solution; the metric can be put in the form

$$d\sigma^2 = g_{ij}\mathrm{d}x^i\,\mathrm{d}x^j = e^{2\beta(r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ . \tag{23}$$

The components of the Ricci tensor for such a metric can be obtained from our calculation of the Schwarzschild metric, where we calculated the Ricci tensor for a spherically symmetric spacetime. By setting $\alpha = 0$ and $\partial_0\beta = 0$ in Equations 14 in the "Schwarzschild" chapter, we get

$$
\begin{aligned}
^{(3)}R_{11} &= \tfrac{2}{r}\partial_1\beta \\
^{(3)}R_{22} &= e^{-2\beta}(r\partial_1\beta - 1) + 1 \\
^{(3)}R_{33} &= [e^{-2\beta}(r\partial_1\beta - 1) + 1]\sin^2\theta \ .
\end{aligned}
\tag{24}
$$

We set these proportional to the metric using Equation 22, and can solve for $\beta(r)$:

$$
\begin{aligned}
R_{11} &= \tfrac{2}{r}\partial_r\beta = 2kg_{11} = 2ke^{2\beta(r)} \\
R_{22} &= e^{-2\beta}(r\partial_r\beta - 1) + 1 = 2kg_{22} = 2kr^2
\end{aligned}
\tag{25}
$$

by simple algebra, one can eliminate $\partial_r \beta$ and obtain

$$\beta = -\frac{1}{2}\ln(1 - kr^2) \ . \tag{26}$$

This gives us the following metric on spacetime:

$$ds^2 = -dt^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \ . \tag{27}$$

This is the **Robertson-Walker metric**. Note that we have not yet made use of Einstein's equations; those will determine the behavior of the scale factor $a(t)$.

Furthermore, we can substitute

$$\begin{aligned}
k &\rightarrow \frac{k}{|k|}\\
r &\rightarrow \sqrt{|k|}\,r\\
a &\rightarrow \frac{a}{\sqrt{|k|}}
\end{aligned} \tag{28}$$

and leave Equation 27 invariant. Therefore the only relevant parameter is $k/|k|$, and there are three cases of interest: $k = -1$, $k = 0$, and $k = +1$. The $k = -1$ case corresponds to constant negative curvature on Σ, and is called **open**; the $k = 0$ case corresponds to no curvature on Σ, and is called **flat**; the $k = +1$ case corresponds to positive curvature on Σ, and is called **closed**.

Let us have a closer look at the three cases.

1. For the flat case $k = 0$ the metric on Σ is

$$\begin{aligned}
d\sigma^2 &= \mathrm{d}r^2 + r^2 d\Omega^2\\
&= \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \ ,
\end{aligned} \tag{29}$$

which is simply flat Euclidean space. Globally, it could describe $\mathbf{R}^3$ or a more complicated manifold, such as the three-torus $S^1 \times S^1 \times S^1$.

2. For the closed case $k = +1$ we can define $r = \sin\chi$ to write the metric on Σ as

$$d\sigma^2 = \mathrm{d}\chi^2 + \sin^2\chi\,d\Omega^2 \ , \tag{30}$$

which is the metric of a three-sphere. In this case the only possible global structure is actually the three-sphere (except for the non-orientable manifold $\mathbf{R}\mathrm{P}^3$). This space is known as **de Sitter space**.

3. Finally in the open $k = -1$ case we can set $r = \sinh \psi$ to obtain

$$d\sigma^2 = \mathrm{d}\psi^2 + \sinh^2 \psi \, d\Omega^2 \ . \tag{31}$$

This is the metric for a three-dimensional space of constant negative curvature; it is hard to visualize, but think of the saddle example we discussed when we introduced curvature. Globally such a space could extend forever (which is the origin of the word "open"), but it could also describe a non-simply-connected compact space (so "open" is really not the most accurate description). This space is known as **anti de Sitter space**.

4. Friedmann equations

The Robertson-Walker metric is the most general metric for which the spatial part is maximally symmetric (homogeneous and isotropic), and is thus consistent with the cosmological principle. However, as you surely have noticed, so far we did not attempt to solve Einstein's equation. The solution to Einstein's equation subject to the restrictions set by the Robertson-Walker metric, will tell us the evolution of the scale factor, $a(t)$.

We proceed as follows. First, we compute the connection coefficients and the curvature tensor of the Robertson-Walker metric. Setting $\dot{a} \equiv da/dt$, the Christoffel symbols are given by

$$
\begin{aligned}
\Gamma^0_{11} &= \frac{a\dot{a}}{1-kr^2} \qquad \Gamma^0_{22} = a\dot{a}r^2 \qquad \Gamma^0_{33} = a\dot{a}r^2 \sin^2\theta \\
\Gamma^1_{01} &= \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a} \\
\Gamma^1_{11} &= \frac{kr}{1-kr^2} \\
\Gamma^1_{22} &= -r(1-kr^2) \qquad \Gamma^1_{33} = -r(1-kr^2)\sin^2\theta \\
\Gamma^2_{12} &= \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \\
\Gamma^2_{33} &= -\sin\theta\cos\theta \qquad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \ .
\end{aligned}
\tag{32}
$$

The nonzero components of the Ricci tensor are

$$
\begin{aligned}
R_{00} &= -3\frac{\ddot{a}}{a} \\
R_{11} &= \frac{a\ddot{a}+2\dot{a}^2+2k}{1-kr^2} \\
R_{22} &= r^2(a\ddot{a}+2\dot{a}^2+2k) \\
R_{33} &= r^2(a\ddot{a}+2\dot{a}^2+2k)\sin^2\theta \ ,
\end{aligned}
\tag{33}
$$

and the Ricci scalar is then

$$R = \frac{6}{a^2}(a\ddot{a}+\dot{a}^2+k) \ . \tag{34}$$

Since the universe is not empty, we are not interested in vacuum solutions to Einstein's Equation. Instead, we consider a model in which the matter and energy in the universe are described by a perfect fluid. This is of course in accordance with the cosmological principle.

Recall that perfect fluids were defined as fluids which are isotropic in their rest frame. For such fluids, the energy-momentum tensor is written as

$$T_{\mu\nu} = (p + \rho)U_\mu U_\nu + p g_{\mu\nu} , \tag{35}$$

where ρ and p are the energy density and pressure (respectively) as measured in the rest frame, and U^μ is the four-velocity of the fluid. It is clear that, if a fluid which is isotropic in some frame leads to a metric which is isotropic in some frame, the two frames will coincide; that is, the fluid will be at rest in comoving coordinates. In these coordinates, the four-velocity is then

$$U^\mu = (1, 0, 0, 0) , \tag{36}$$

and the energy-momentum tensor is

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix} . \tag{37}$$

With one index raised this takes the more convenient form

$$T^\mu{}_\nu = \mathrm{diag}(-\rho, p, p, p) . \tag{38}$$

Note that the trace is given by

$$T = T^\mu{}_\mu = -\rho + 3p . \tag{39}$$

We can now plug it in Einstein's equations,

$$R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) . \tag{40}$$

The $\mu\nu = 00$ equation is

$$\begin{aligned} & -3\frac{\ddot{a}}{a} = 8\pi G \left(\rho + \tfrac{1}{2}(-\rho + 3p) \right) \\ \Rightarrow \quad & -3\frac{\ddot{a}}{a} = 4\pi G(\rho + 3p) . \end{aligned} \tag{41}$$

The $\mu\nu = ij$ equations become

$$\begin{aligned} R_{ij} &= 8\pi G \left(g_{ij}p - \tfrac{1}{2}g_{ij}(-\rho + 3p) \right) \\ &= 8\pi G \left(\tfrac{1}{2}g_{ij}\rho - \tfrac{1}{2}g_{ij}p \right) \\ &= 4\pi G g_{ij}(\rho - p) \end{aligned} \tag{42}$$

Note that the 9 ($\mu\nu = ij$) equations are in fact reduced to one distinct equation, due to isotropy. Thus we can chose $\mu\nu = 11$, and get (using $g_{11} = a^2(t)/(1 - kr^2)$),

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 4\pi G(\rho - p) \ . \tag{43}$$

We can use Equation 41 to eliminate second derivatives in Equation 43, and do a little cleaning up to obtain

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \ , \tag{44}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \ . \tag{45}$$

Together, Equations 44 and Equation 45 are known as the **Friedmann equations**, and metrics of the form (Equation 27) which obey these equations define Friedmann-Robertson-Walker (FRW) universes.

5. Global properties of the universe

The second of Friedmann equations (equation 45) can be used to infer global properties of the universe.

We first define the **Hubble parameter**, which characterizes the rate of expansion by

$$H \equiv \frac{\dot{a}}{a} \ . \tag{46}$$

Note that we have to divide $\dot{a}$ by a to get a measurable quantity, since the overall scale of a is irrelevant. We further define the **deceleration parameter**,

$$q = -\frac{a\ddot{a}}{\dot{a}^2} \ , \tag{47}$$

which measures the rate of change of the rate of expansion.

Hubble's parameter (**which is time dependent!**) at present day epoch is called **Hubble's constant**,

$$H_0 \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_0} = 70 \pm 3 \, \text{km} \, \text{s}^{-1} \, \text{Mpc}^{-1} \tag{48}$$

where "Mpc" stands for "megaparsec", which is 3×10^{24} cm. The value quoted is based on 7 year data release of the WMAP satellite in 2010. Hubble's constant is extensively measured since Hubble's days (and you will understand shortly why its exact value is so

important !). The recent measurement from the "Planck" mission released in 2018 is $H_0 = 67.66 \pm 0.42 \, \mathrm{km \, s^{-1} \, Mpc^{-1}}$,

With Hubble's parameter, we can write Equation 45 as

$$1 = \frac{8\pi G}{3H^2}\rho - \frac{k}{a^2 H^2} \tag{49}$$

We can now define a quantity known as the **critical density**

$$\rho_{\mathrm{crit}} = \frac{3H^2}{8\pi G} \ . \tag{50}$$

which at present day epoch takes the value

$$\rho_{\mathrm{crit},0} = \frac{3H_0^2}{8\pi G} = \left\{ \begin{array}{ll} 9.21 \times 10^{-30} & \mathrm{g \, cm^{-3}} \\ 5.19 \times 10^3 & \mathrm{eV \, cm^{-3}} \end{array} \right. \tag{51}$$

We can further define the **density parameter**,

$$\Omega = \frac{8\pi G}{3H^2}\rho = \frac{\rho}{\rho_{\mathrm{crit}}} \ , \tag{52}$$

We can now understand the meaning of the term "critical" density (which changes with time !), as we can write the Friedman Equation (equation 45, 49) as

$$\Omega - 1 = \frac{k}{H^2 a^2} \ . \tag{53}$$

The sign of k is therefore determined by whether Ω is greater than, equal to, or less than one. We have

$$\begin{array}{ccccccc} \rho < \rho_{\mathrm{crit}} & \leftrightarrow & \Omega < 1 & \leftrightarrow & k = -1 & \leftrightarrow & \mathrm{open} \\ \rho = \rho_{\mathrm{crit}} & \leftrightarrow & \Omega = 1 & \leftrightarrow & k = 0 & \leftrightarrow & \mathrm{flat} \\ \rho > \rho_{\mathrm{crit}} & \leftrightarrow & \Omega > 1 & \leftrightarrow & k = +1 & \leftrightarrow & \mathrm{closed} \ . \end{array}$$

The density parameter, then, tells us which of the three Robertson-Walker geometries describes our universe. Determining it observationally is an area of intense investigation.

6. Content of the universe

Before we can continue to discuss the evolution of the universe, let us first examine the content of the universe.

Consider the stress-energy tensor (Equation 38), and look at the zero component of the conservation of energy equation:

$$
\begin{aligned}
0 &= \nabla_\mu T^\mu{}_0 \\
&= \partial_\mu T^\mu{}_0 + \Gamma^\mu_{\mu 0} T^0{}_0 - \Gamma^\lambda_{\mu 0} T^\mu{}_\lambda \\
&= -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p) \ .
\end{aligned}
\tag{54}
$$

(where we have used the affine connections calculated in Equation 32).

To make progress it is necessary to choose an **equation of state**, a relationship between ρ and p. Essentially all of the perfect fluids relevant to cosmology obey the simple equation of state

$$
p = w\rho \ ,
\tag{55}
$$

where w is a constant independent of time. The conservation of energy equation becomes

$$
\frac{\dot{\rho}}{\rho} = -3(1+w)\frac{\dot{a}}{a} \ ,
\tag{56}
$$

which can be integrated to obtain

$$
\rho \propto a^{-3(1+w)} \ .
\tag{57}
$$

The two most popular examples of cosmological fluids are known as **dust** and **radiation**. Dust is collisionless, nonrelativistic matter, which zero pressure; hence $w = 0$. Examples include ordinary stars and galaxies, for which the pressure is negligible in comparison with the energy density. Dust is also known as "matter", and universes whose energy density is mostly due to dust are known as **matter-dominated**. The energy density in matter falls off as

$$
\rho_{\mathrm{m}} \propto a^{-3} \ .
\tag{58}
$$

This is simply interpreted as the decrease in the number density of particles as the universe expands. (For dust the energy density is dominated by the rest energy, which is proportional to the number density.)

"Radiation" may be used to describe either actual electromagnetic radiation, or massive particles moving at relative velocities sufficiently close to the speed of light that they become indistinguishable from photons (at least as far as their equation of state is concerned). There are various ways to derive the equation of state for radiation. One is to note that $T_{\mu\nu}$ can be expressed in terms of the EM field strength as

$$
T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F^\nu{}_\lambda - \frac{1}{4}g^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma}\right) \ .
\tag{59}
$$

The trace of this is given by

$$T^{\mu}{}_{\mu} = \frac{1}{4\pi}\left[F^{\mu\lambda}F_{\mu\lambda} - \frac{1}{4}(4)F^{\lambda\sigma}F_{\lambda\sigma}\right] = 0 . \tag{60}$$

Thus, the trace of stress energy tensor is 0; putting this in Equation 39 ($T = -\rho + 3p$), the equation of state for radiation is

$$p = \frac{1}{3}\rho . \tag{61}$$

This result could be derived in more than one way, of course. An alternative derivation is to recall that the momentum of photons / relativistic particles is related to their energy by $E = \vec{p}c$. The pressure exerted by N molecules of relativistic gas in a 3-d container of volume V on a wall is

$$p = \frac{1}{3}\frac{N}{V}\langle \vec{p}\cdot\vec{v}\rangle = \frac{1}{3}\frac{N}{V}\langle E\rangle = \frac{\rho}{3}.$$

A universe in which most of the energy density is in the form of radiation is known as **radiation-dominated**. The energy density in radiation falls off as

$$\rho_{\rm R} \propto a^{-4} . \tag{62}$$

Thus, *the energy density in radiation falls off slightly faster than that in matter*; this is because the number density of photons decreases in the same way as the number density of nonrelativistic particles, but individual photons also lose energy as a^{-1} as they redshift, as we will see later. (Likewise, massive but relativistic particles will lose energy as they "slow down" in comoving coordinates.) We believe that today the energy density of the matter / radiation in the universe is dominated by matter, with $\rho_{\rm mat}/\rho_{\rm rad} \sim 10^3$. However, in the past the universe was much smaller, and the energy density in radiation would have dominated at very early times.

The final form of energy-momentum which we should consider is that of the vacuum itself: this is the addition of the cosmological constant in Einstein's equation. Recall that Einstein's equations with a cosmological constant are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu} , \tag{63}$$

which is clearly the same form as the equations with no cosmological constant but an energy-momentum tensor for the vacuum,

$$T^{\rm (vac)}_{\mu\nu} = -\frac{\Lambda}{8\pi G}g_{\mu\nu} , \tag{64}$$

as Einstein's equation reads $G_{\mu\nu} = 8\pi G(T_{\mu\nu} + T^{\rm (vac)}_{\mu\nu})$. Neglecting any term but the vacuum itself, this has the form of a perfect fluid with

$$\rho = -p = \frac{\Lambda}{8\pi G} . \tag{65}$$

We therefore have $w = -1$, and the energy density is independent of a,

$$\rho_\Lambda \propto a^0. \tag{66}$$

This is what we would expect for the energy density of the vacuum. Since the energy density in matter and radiation decreases as the universe expands, if there is a nonzero vacuum energy it tends to win out over the long term (as long as the universe doesn't start contracting). If this happens, we say that the universe becomes **vacuum-dominated**.

Current measurements (based on 2018 data released from "Planck" mission) give: $\Omega_\Lambda = 0.6911$, $\Omega_{CDM} = 0.3089$, where "CDM" stands for "cold dark matter", out of which the observed baryons (radiating matter) is $\Omega_b = 0.0486$, and Ω_{rad} is negligible.

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