Cosmology: Part II

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This part of the course is based on Refs. [1] - [4].

1. Asymptotic behavior of the universe

The Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \quad . \tag{1}$$

is the most general metric that describes a universe which is spatially homogeneous and isotropic, namely its spatial term is **maximally symmetric**. The curvature constant k can get three values, $k = -1, 0, +1$ describing an **open, flat and closed** universes, respectively.

The **scale factor** $a(t)$ is obtained by using this metric in Einstein's equation,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{2}$$

The results are two equations,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \ , \tag{3}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \ , \tag{4}$$

which are known together as **Friedmann Equations**. These equations describe the evolution of the scale factor, hence of the universe as a whole.

1.1. The Big Bang

While it is possible to solve the Friedmann equations exactly in various simple cases, it is often more useful to know the qualitative behavior of various possibilities.

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Consider first a universe with no cosmological constant: $\Lambda = 0$. Consider the behavior of universes filled with fluids of positive energy ($\rho > 0$) and nonnegative pressure ($p \geq 0$). By the first of Friedmann's equations (Equation 3) we must have $\ddot{a} < 0$. Since we know from observations of distant galaxies that the universe is expanding ($\dot{a} > 0$), this means that the universe is "decelerating" namely, the expansion rate is decreasing.

This is what we should expect, since the gravitational attraction of the matter in the universe works against the expansion. The fact that the universe can only decelerate means that it must have been expanding even faster in the past; if we trace the evolution backwards in time, we necessarily reach a singularity at $a = 0$. Notice that if $\ddot{a}$ were exactly zero, $a(t)$ would be a straight line, and the age of the universe would be H_0^{-1}. Since $\ddot{a}$ is actually negative, the universe must be somewhat younger than that. This is demonstrated in Figure 1.

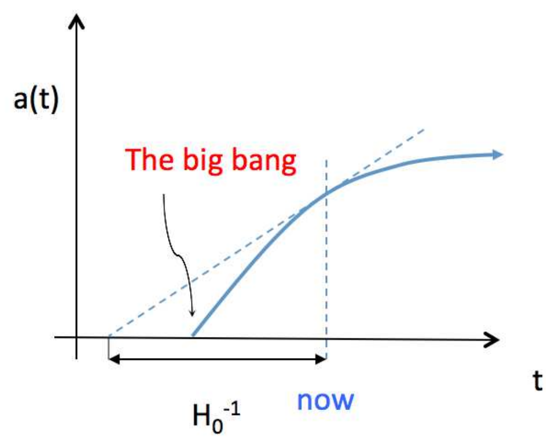

Fig. 1.— In a universe with no cosmological constant, we know that it is expanding ($\dot{a} > 0$) and decelerating ($\ddot{a} < 0$). Thus, there must have been a point in the past when $a = 0$. This is the Big Bang.

This singularity at $a = 0$ is the **Big Bang**. It represents the creation of the universe from a singular state, **not** explosion of matter into a pre-existing spacetime.

It might be hoped that the perfect symmetry of our FRW universes was responsible for this singularity, but in fact it's not true; the singularity theorems predict that any universe with $\rho > 0$ and $p \geq 0$ must have begun at a singularity. Of course the energy density becomes arbitrarily high as $a \to 0$, and we don't expect classical general relativity to be an accurate description of nature in this regime; hopefully a consistent theory of quantum gravity will be able to fix things up.

1.2. Future evolution: open and flat universes

The future evolution is different for different values of k. For the open and flat cases, $k \leq 0$, The second of Friedmann's equations, Equation 4 implies

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 + |k| \ . \tag{5}$$

The right hand side is *strictly* positive (since we are assuming $\rho > 0$), so $\dot{a}$ never passes through zero. Since we know that today $\dot{a} > 0$, it must be *positive at all time*. Thus, **the open and flat universes expand forever** — they are temporally as well as spatially open. (Please keep in mind what assumptions go into this — namely, that there is a nonzero positive energy density, $\rho > 0$. Negative energy density universes do not have to expand forever, even if they are "open".)

How fast do these universes keep expanding? Consider the quantity ρa^3 (which is constant in matter-dominated universes). Recall that we wrote the energy conservation equation as

$$0 = \nabla_\mu T^\mu{}_0 = -\partial_0 \rho - 3\frac{\dot{a}}{a}(\rho + p) \ . \tag{6}$$

By simple algebra, we can write this equation in the form

$$\frac{d}{dt}(\rho a^3) = -3a^2 \dot{a} p \ . \tag{7}$$

The right hand side is either zero or negative; therefore

$$\frac{d}{dt}(\rho a^3) \leq 0 \ . \tag{8}$$

Thus, in an ever-expanding universe, where $a \to \infty$ Equation 8 implies that ρa^2 must go to zero in the limit $a \to \infty$. From Equation 5 we thus find that in this limit

$$\dot{a}^2 \to |k| \ . \tag{9}$$

(Remember that this is true for $k \leq 0$.) Thus, for $k = -1$ the expansion approaches the limiting value $\dot{a} \to 1$, while for $k = 0$ the universe keeps expanding, but more and more slowly (note that any power law expansion of the form $a(t) \propto t^\alpha$ with $0 \leq \alpha < 1$ results in an asymptotic limit $\dot{a} \to 0$).

1.3. Future evolution: closed universes

For the closed universes ($k = +1$), Equation 4 becomes

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - 1 \ . \tag{10}$$

The argument that $\rho a^2 \to 0$ as $a \to \infty$ still applies; but in that case, the right hand side of Equation 10 would become negative, which can't happen. Therefore the universe does not expand indefinitely; a possesses an upper bound $a_{\max}$. As a approaches $a_{\max}$, the first of Friedmann's equations (Equation 3) implies

$$\ddot{a} \to -\frac{4\pi G}{3}(\rho + 3p)a_{\max} < 0 \ . \tag{11}$$

Thus $\ddot{a}$ is finite and negative at this point, so a reaches $a_{\max}$ and starts decreasing, whereupon (since $\ddot{a} < 0$) it will inevitably continue to contract to zero — the **Big Crunch**. Thus, the closed universes (again, under our assumptions of positive ρ and nonnegative p) are closed in time as well as space (see Figure 2).

Note that a universe for which $\Omega > 1$ is necessarily closed (see "Cosmology", part I, Equation 53), while a universe for which $\Omega <$ is open. In Figure 2, plotted is the evolution of the scale factor (or the average distance between the galaxies) for different values of Ω_m and $\Omega_v \equiv \Omega_\Lambda$, where $k = 0$ and $\Omega_R = 0$ are taken. For $\Omega = \Omega_m + \Omega_V > 1$ (yellow) the universe is closed; for $\Omega = 1$ (green) it is flat, while for $\Omega < 1$ (blue) it is open. Current observations suggest $\Omega_V = 0.7$ and $\Omega_m = 0.3$ (red), in which case the universe accelerates- see below.

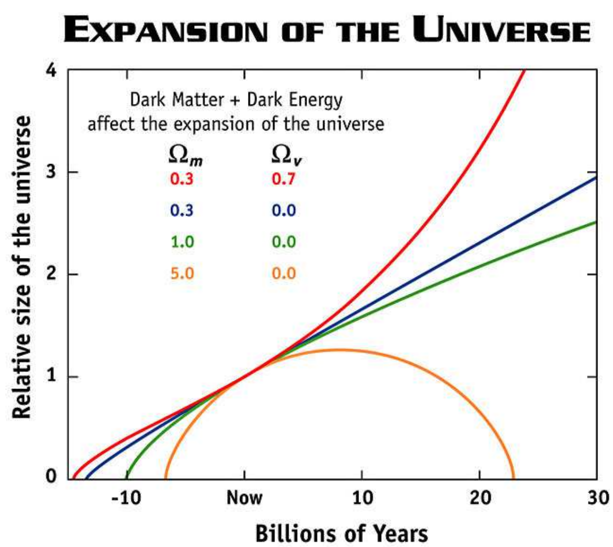

Fig. 2.— The fate of the universe depends on its curvature, and the values of the density parameters. A close universe ($k = 1$) will end in a "crunch", while an open or flat universes ($k = -1, 0$) will expand forever. Current observations suggest that the universe does not only expands, but accelerates - see further discussion below.

It is often common to define a fictitious energy density associated with the spatial curvature by

$$\rho_c \equiv -\frac{3k}{8\pi G a^2} \tag{12}$$

with a corresponding density parameter

$$\Omega_c = -\frac{k}{H^2 a^2}. \tag{13}$$

Using these variables, Friedmann Equation (Equation 4) is written as

$$H^2 = \frac{8\pi G}{3} \sum_i \rho_i \tag{14}$$

where $\rho_i \equiv \rho_m\ ,\ \rho_R\ ,\ \rho_\Lambda\ ,\ \rho_c$. Alternatively, dividing both sides by H^2 this equation is written as

$$1 = \sum_i \Omega_i. \tag{15}$$

Note that by no means ρ_i must all be non-negative. While matter and radiation arise from dynamical particles and fields, and so we do expect that ρ_m and ρ_R be positive, we cannot say anything about the vacuum and the curvature, ρ_Λ and ρ_c.

2. The cosmological redshift

Because of the time dependence of the scale factor $a(t)$, the FRW metric is not static. Since $a(t)$ multiplies the spatial coordinates, any proper distance $l(t)$ will change with time in proportion to $a(t)$:

$$l(t) = l_0 a(t) \propto a(t)\ , \tag{16}$$

where l_0 is the comoving distance between two (static) observers. In particular, the proper separation between any two observers, located at constant comoving coordinates, will change with time. Let the coordinate separation between two such observers (lets say, located at nearby galaxies) be δr, so that the proper separation is $\delta l = a(t)\delta r$. Each of the two observers will attribute to the other a velocity

$$\delta v = \frac{d}{dt}\delta l = \dot{a}\delta r = \left(\frac{\dot{a}}{a}\right)\delta l \tag{17}$$

This leads to several important physical consequences of rather generic nature.

Consider a narrow pencil of electromagnetic radiation which crosses any two comoving observers, separated by proper distance δl. The transit time is $\delta t = \delta l/c$. Let the frequency of radiation measured by the first observer be ω. Since the first observer sees the second one receding with velocity δv, he expected the second observer to measure a Doppler shifted frequency $(\omega + \delta\omega)$ where

$$\frac{\delta\omega}{\omega} = -\frac{\delta v}{c} = -\left(\frac{\dot{a}}{a}\right)\frac{\delta l}{c} = -\left(\frac{\dot{a}}{a}\right)\delta t = -\frac{\delta a}{a} \tag{18}$$

(we have assumed that the two observers are separated by infinitesimal distance of first order δl, therefore we could introduced locally inertial frame encompassing both observers; the laws of special relativity can be applied in this frame).

Equation 18 can be integrated to give

$$\omega(t)a(t) = \text{constant} \ . \tag{19}$$

In other words, the frequency of electromagnetic radiation changes due to the expansion of the universe according to the law $\omega \propto a^{-1}$. Note that we have made implicit use of the homogeneity of the spacetime in extending the local result to a global context.

Thus, a photon emitted with frequency ω_1 will be observed at some later time with a lower frequency ω_0 as the universe expands:

$$\frac{\omega_0}{\omega_1} = \frac{a_1}{a_0} \ . \tag{20}$$

Cosmologists like to speak of this in terms of the **redshift** z between the two events, defined by the fractional change in wavelength:

$$\begin{aligned} z &= \tfrac{\lambda_0 - \lambda_1}{\lambda_1} \\ &= \tfrac{a_0}{a_1} - 1 \ . \end{aligned} \tag{21}$$

Notice that this redshift is **not** the same as the conventional Doppler effect; it is the expansion of space, not the relative velocities of the observer and emitter, which leads to the redshift.

The redshift is something we can measure; we know the rest-frame wavelengths of various spectral lines in the radiation from distant galaxies, so we can tell how much their wavelengths have changed along the path from time t_1 when they were emitted to time t_0 when they were observed. We therefore know the ratio of the scale factors at these two times. But we don't know the times themselves; the photons are not clever enough to tell us how much coordinate time has elapsed on their journey. We have to work harder to extract this information.

However, using the definition of the redshift in Equation 21 with $a_0 \equiv a(t = t_0)$ is the scale factor *today*, the redshift z is often used to replace the time coordinate t or the value of the scaling factor $a(t)$ at that time.

The same argument holds when considering the motion of massive particles. Consider again two comoving observers separated by proper distance δl. Let a massive particle pass the first observer with velocity v. When the particle crossed the proper distance δl (in time interval δt), it passed the second observer, whose velocity (relative to the first one) is

$$\delta u = \frac{\dot{a}}{a}\delta l = \frac{\dot{a}}{a} v\delta t = v\frac{\delta a}{a} \tag{22}$$

The second observer will attribute to the particle the velocity

$$v' = \frac{v - \delta u}{1 - v\delta u} = v - (1 - v^2)\delta u + \mathcal{O}[(\delta u)^2] = v - (1 - v^2)v\frac{\delta a}{a}, \tag{23}$$

where we have used the special-relativistic formula for addition of velocities, which is valid in the infinitesimal regime around the first observer. Rewriting this equation as

$$\delta v = -v(1 - v^2)\frac{\delta a}{a} \tag{24}$$

and integrating, we get

$$p = \frac{v}{\sqrt{1 - v^2}} = \frac{\text{constant}}{a} \tag{25}$$

In other words, the magnitude of the 3-momentum decreases as a^{-1} due to the expansion. For non-relativistic particles, $v \propto p$ and the velocity itself decays as a^{-1}. The particle therefore "slows down" with respect to the comoving coordinates as the universe expands. This is an actual slowing down, in the sense that a gas of particles with initially high relative velocities will cool down as the universe expands.

3. Distance measurement in the universe

Roughly speaking, since a photon moves at the speed of light its travel time should simply be its distance. But what is the "distance" of a far away galaxy in an expanding universe? The comoving distance is not especially useful, since it is not measurable, and furthermore because the galaxies need not be comoving in general. Instead we can define the **luminosity distance** as

$$d_L^2 = \frac{L}{4\pi F} \ , \tag{26}$$

where L is the absolute luminosity of the source and F is the flux measured by the observer (the energy per unit time per unit area of some detector). The definition comes from the fact that in flat space, for a source at distance d the flux over the luminosity is just one over the area of a sphere centered around the source, $F/L = 1/A(d) = 1/4\pi d^2$.

In an FRW universe, however, *the flux will be diluted.* Conservation of photons tells us that the total number of photons emitted by the source will eventually pass through a sphere at comoving distance r from the emitter. Such a sphere is at a physical distance $d = a_0 r$, where a_0 is the scale factor when the photons are observed.

But the flux is diluted by two additional effects: the individual photons redshift by a factor $(1+z)$ (=their energy is decreased), and the photons hit the sphere less frequently, since two photons emitted a time δt apart will be measured at a time $(1+z)\delta t$ apart. Therefore we will have

$$\frac{F}{L} = \frac{1}{4\pi a_0^2 r^2 (1+z)^2} \ , \tag{27}$$

or

$$d_L = a_0 r(1+z) \ . \tag{28}$$

The luminosity distance d_L is something we might hope to measure, since there are some astrophysical sources whose absolute luminosities are known ("standard candles"). But r is not observable, so we have to remove that from our equation.

Before we do that, we note that another observable parameter for distant sources is the **angular diameter**. Consider a distant object of physical size D (say, a distant galaxy) which emits photons at comoving time t_1. These photons are observed at time t_0. Assume that the object subtends an angle δ to the observer, then, for small δ, we have $D = ra(t_1)\delta$. The 'angular diameter distance', $d_A(z)$ for the source is defined via the relation $\delta = (D/d_A)$; so, we find that

$$d_A(z) = ra(t_1) = a_0 r(t_1)(1+z)^{-1} \ . \tag{29}$$

Clearly, $d_L = (1+z)^2 d_A$.

Let us return to the problem of expressing d_L (and d_A) in terms of measurable quantities, namely that do not contain r, which is not measurable. From the FRW metric on a null geodesic (chosen to be radial for convenience) we have

$$0 = ds^2 = -\mathrm{d}t^2 + \frac{a^2}{1-kr^2}\mathrm{d}r^2 \ , \tag{30}$$

or

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_0^{r_1} \frac{dr}{(1-kr^2)^{1/2}} = \begin{cases} \sin^{-1}(r_1) \ ; & k=+1, \\ r_1 \ ; & k=0, \\ \sinh^{-1}(r_1); & k=-1. \end{cases} \ . \tag{31}$$

For galaxies not too far away, we can expand the scale factor in a Taylor series about its present value:

$$\begin{aligned} a(t_1) &= a_0 + (\dot{a})_0(t_1-t_0) + \tfrac{1}{2}(\ddot{a})_0(t_1-t_0)^2 + \ldots \\ &= a_0\left[1 + \left(\tfrac{\dot{a}}{a}\right)_0(t_1-t_0) + \tfrac{1}{2}\left(\tfrac{\ddot{a}}{a}\right)_0(t_1-t_0)^2 + \ldots\right] \\ &= a_0\left[1 + H_0(t_1-t_0) - \tfrac{1}{2}q_0 H_0^2(t_1-t_0)^2 + \ldots\right] \ , \end{aligned} \tag{32}$$

where we have used the definition of Hubble's constant and the deceleration parameter,

$$H_0 \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_0} \quad ; \qquad q_0 \equiv -\left(\frac{a\ddot{a}}{\dot{a}^2}\right)_{t=t_0} . \tag{33}$$

We can now use Taylor expansion in both sides of Equation 31 (Keeping only the leading term in the right hand side, and use Equation 32 on the left hand side), to find

$$r = a_0^{-1}\left[(t_0 - t_1) + \frac{1}{2}H_0(t_0 - t_1)^2 + \ldots\right] . \tag{34}$$

Recalling that $(1+z) = a_0/a(t_1)$, Equation 32 takes the form

$$\frac{1}{1+z} = 1 + H_0(t_1 - t_0) - \frac{1}{2}q_0 H_0^2 (t_1 - t_0)^2 + \ldots . \tag{35}$$

For small $H_0(t_1 - t_0)$ this can be inverted to yield

$$t_0 - t_1 = H_0^{-1}\left[z - \left(1 + \frac{q_0}{2}\right)z^2 + \ldots\right] . \tag{36}$$

Substituting this back again into Equation 34 gives

$$r = \frac{1}{a_0 H_0}\left[z - \frac{1}{2}\left(1 + q_0\right)z^2 + \ldots\right] . \tag{37}$$

Finally, using this in Equation 28 ($d_L = a_0 r(1+z)$), yields **Hubble's Law**:

$$d_L = H_0^{-1}\left[z + \frac{1}{2}(1 - q_0)z^2 + \ldots\right] . \tag{38}$$

Therefore, measurement of the luminosity distances and redshifts of a sufficient number of galaxies allows us to determine H_0 and q_0, and therefore takes us a long way to deciding what kind of FRW universe we live in.

4. Evolution of the scale factor $a(t)$

We can solve Friedmann's Equation (Equation 4) and obtain the evolution of $a(t)$ in the various scenarios. Since the universe is composed of matter, radiation and vacuum energy, the evolution depends on answering two questions: (i) whether the universe is flat, open or closed; and (ii) what is the dominant energy content of the universe.

In Cosmology part I, we showed that for all three ingredients (matter, radiation and vacuum energy), we can write the equation of state in the form $p = \omega\rho$, from which the conservation of energy becomes

$$p = \omega\rho \Rightarrow \rho \propto a^{-3(1+\omega)} \Rightarrow \begin{cases} \rho_m \propto a^{-3} & [\omega = 0] \\ \rho_r \propto a^{-4} & [\omega = 1/3] \\ \rho_\Lambda \propto a^0 & [\omega = -1] \end{cases} \tag{39}$$

Where ρ_r is the energy density in both radiation and relativistic matter, ρ_m is the energy density in non-relativistic matter and ρ_Λ is the vacuum energy density. Thus,

$$\rho_m(t) = \rho_m(t_0)\left(\frac{a_0}{a}\right)^3 = \rho_c\Omega_m(1+z)^3 \tag{40}$$

where $\Omega_m \equiv \rho_{m,0}/\rho_c$; $\rho_c = \rho_{crit,0} = 3H_0^2/8\pi G$ is the critical density; and the subscript $_0$ represent present time values.

Similarly,

$$\rho_r(t) = \rho_r(t_0)\left(\frac{a_0}{a}\right)^4 = \rho_c\Omega_r(1+z)^4, \tag{41}$$

and

$$\rho_\Lambda(t) = \rho_\Lambda(t_0). \tag{42}$$

Observations suggest that at present epoch

$$\Omega_{\rm total} \equiv \Omega \simeq 1; \quad \Omega_m \sim 0.3 \quad \Omega_r \simeq 8.2 \times 10^{-5} \tag{43}$$

Thus, at present, matter dominates over radiation (and vacuum over both). But when looking into equations 40, 41 and 42, clearly when looking at the past (z increases), radiation energy density grows faster than matter (and vacuum) energy densities as we go to earlier phases of the universe. At some time, $t = t_{eq}$ in the past (corresponding to a value $a = a_{eq}$ and redshift $z = z_{eq}$) the radiation and matter have had equal energy densities. From Equations 40, 41 and 43 we get

$$(1+z_{eq}) = \frac{a_0}{a_{eq}} = \frac{\Omega_m}{\Omega_r} \simeq 3.6 \times 10^3 \tag{44}$$

Since the temperature of the radiation grows as a^{-1} (recall Stefan-Boltzmann's law, $T^4 \propto \rho_r \propto (1+z)^4$), the temperature of the universe at this epoch was

$$T_{eq} = T_0(1+z_{eq}) = 2.7 \times 5.8 \times 10^3 \simeq 1.0 \times 10^{4\circ}\text{K} = 0.9 \text{ eV} \tag{45}$$

where $T_0 = 2.7^\circ$ K is the temperature of the CMB radiation.

We can now solve Friedmann Equation (Equation 4) for various geometries and contents of the universe. Let us focus on the flat universe ($k = 0$). For matter dominated universe, we find

$$a(t) = \left(\frac{9}{4}\frac{8\pi G}{3}\Omega_m\rho_c a_0^3\right)^{1/3} t^{2/3}, \tag{46}$$

while for radiation-dominated flat universe

$$a(t) = \left(4 \times \frac{8\pi G}{3}\Omega_r \rho_c a_0^4\right)^{1/4} t^{1/2}. \tag{47}$$

Clearly, at very early times, $t \ll t_{eq}$, the energy density of the universe is dominated by radiation, while at later times, $t \gg t_{eq}$ it is dominated by matter. Given that we can consider present day epoch as being matter-dominated, the Equilibrium time, t_{eq} is easily obtained using Equations 44 and 46.

For universe which is dominated by positive vacuum energy density, $\rho_\Lambda = \Lambda/8\pi G$, the solution (for closed, flat and open universes) is

$$a(t) \propto \begin{cases} \sinh\left[\left(\frac{\Lambda}{3}\right)^{1/2} t\right] & (k=-1) \\ \exp\left[\pm\left(\frac{\Lambda}{3}\right)^{1/2} t\right] & (k=0) \\ \cosh\left[\left(\frac{\Lambda}{3}\right)^{1/2} t\right] & (k=+1) \end{cases} \tag{48}$$

All these solutions represent, in fact, the same space-time, just in different coordinates. This spacetime is known as **de Sitter space**, which is maximally symmetric. Solution also exists with $\Lambda < 0$, which is known as **anti de Sitter space**.

5. Observational evidence

As usual in physics, we cannot accept any theory - regardless how mathematically beautiful it may be, without strong experimental or observational support. The “big bang” theory is strongly supported by 5 independent measurements.

1. **Hubble’s law of expansion**. Historically, the first observational evidence (excluding the fact that the skies are dark) was Hubble’s law (Equation 38, discovered in 1929 [1]. From redshift measurements Hubble determined the radial velocities of 24 galaxies, for which he could estimate their distance.

2. **The cosmic microwave background (CMB) radiation**. We will discuss this shortly.

   Briefly, CMB is landmark evidence of the Big Bang origin of the universe. When the universe was young, before the formation of stars and planets, it was denser, much

[1] In fact, it we predicted by Georges Lemaitré 2 years earlier, in 1927

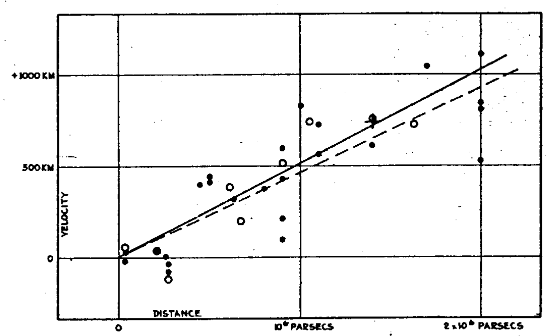

FIGURE 1

Fig. 3.— Hubble's expansion law, as taken from his original paper in 1929. The distance is measured in millions of parsecs ($1pc = 3 \times 10^{18}$ cm) and the velocity in km/s

hotter, and filled with a uniform glow from a white-hot fog of hydrogen plasma. As the universe expanded, both the plasma and the radiation filling it grew cooler. When the universe cooled enough, protons and electrons combined to form neutral hydrogen atoms. Unlike the uncombined protons and electrons, these newly conceived atoms could not scatter the thermal radiation by Thomson scattering, and so the universe became transparent instead of being an opaque fog; this is known in cosmology as **recombination**.

At this stage, the photons decouple the plasma. These photons are propagating in the universe ever since, growing fainter and less energetic, due to the continuous expansion of space. They are isotropic[2] and filled the entire space, having black-body spectrum with temperature today of $T = 2.726\ ^\circ K$. As such, the spectral peak is at 160 GHz, which is in the microwave range of frequencies. These photons are the oldest electromagnetic radiation that can be observed in the universe.

The cosmic microwave background was first predicted in 1948 by Ralph Alpher and Robert Herman. It was discovered in 1964 by Arno Penzias and Robert Wilson, who received the 1978 Nobel prize in physics for their discovery.

3. **Primordial abundances of light elements.** (Another name for that is "big bang nucleosynthesis"). While all the elements heavier than lithium were produced in the cores of stars, the light elements in our universe- namely, hydrogen (^{1}H), deuterium (^{2}D), helium (^{4}He) and its isotope helium-3 (^{3}He) and a very small amount of lithium

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[2]anisotropy is seen at a level of $1 : 10^5$.

7 (^{7}Li) (as well as a few unstable isotopes) were all formed within the first 20 minutes of the big bang.

The theory (see below) provides an excellent prediction for their relative abundances as is observed in the universe. The relative abundances depend on a single parameter, the ratio of photons to baryons. Roughly, the mass fraction of Helium 4 (^{4}He) to hydrogen in the universe is 0.245, and the other rations are

$$\left(\frac{\mathrm{D}}{\mathrm{H}}\right)_\mathrm{p} = 3.6 \times 10^{-5},$$
$$\left(\frac{^3\mathrm{He}}{\mathrm{H}}\right)_\mathrm{p} = 1.2 \times 10^{-5},$$
$$\left(\frac{^7\mathrm{Li}}{\mathrm{H}}\right)_\mathrm{p} = 1.2 \times 10^{-11}.$$

Below I provide some basic description of these calculations.

4. **Galactic evolution and distribution**. Although the universe was initially nearly homogeneous, small fluctuations grew with time to form larger and larger objects - eventually, the galaxies (and clusters of galaxies) we observe. This process is known as "structure formation". A natural outcome is that the structure and morphologies of galaxies evolve over cosmic times, from the time they were "born", until today.

   As populations of stars have been aging and evolving, so that distant galaxies (which are observed as they were in the early universe) appear very different from nearby galaxies (observed in a more recent state). Moreover, galaxies that formed relatively recently, appear markedly different from galaxies formed at similar distances but shortly after the Big Bang. These observations are strong arguments in favor of the big bang model, and against the alternative steady-state model. Observations of star formation, galaxy and quasar distributions and larger structures, agree well with Big Bang simulations of the formation of structure in the universe, and are helping to complete details of the theory.

5. **Baryon acoustic oscillations (BAO).** These are fluctuations in the density of the visible baryonic matter (normal matter) of the universe, caused by acoustic density waves in the primordial plasma of the early universe. These oscillations provide a "standard ruler" for length scale, which is given by the maximum distance the acoustic waves could travel in the primordial plasma before the plasma cooled to the point where it became neutral atoms (the epoch of recombination). At this point, the expansion of the plasma density waves stopped, "freezing" them into place. The length of this standard ruler ($\approx$ 490 million light years in today's universe) can be measured by looking at the large scale structure of matter using astronomical surveys.

These oscillations are imprinted on the CMB as small ($\sim 10^{-5}$) fluctuations in the observed temperature. These fluctuations are not entirely random, but are correlated between different angles in space (see below, if we have time). They appear as location of peaks when looking at the power spectrum of CMB fluctuations. The location and relative height of the peaks are determined by the cosmological parameters, and provide an independent measure of their values.

6. Thermal history of the Universe

As stated above, one of the triumphs of modern cosmology is its ability to predict the relative abundances of light elements in the universe, as well as the different structures that are observed when measuring the fluctuations within the cosmic microwave background (CMB); these fluctuations indicate relative abundances of matter.

By simply applying the rates for nuclear reactions that are measured in the laboratory to a plasma of the correct baryon density in an expanding universe roughly a few seconds to minutes after the big bang (at temperatures $T \sim 0.1 - 10$ MeV), we find that the neutrons and protons in the universe organize themselves into roughly 75% hydrogen and 25% helium (by mass), with calculable trace abundances of deuterium and ^{7}Li. The predictions are in excellent agreement with the observations, and the success of the theory allows us to place important constraints to the content and evolution of the universe just seconds after the big bang.

Going back in time, when the redshift was higher than $z_{eq} \sim 3.6 \times 10^3$, the universe was dominated by radiation. In the radiation dominated phase of the universe, its temperature was $T > T_{eq} \sim 1$ eV, and it changed with time according to $T \propto a^{-1} \propto (1 + z)$.

At these times, the content of the universe was different than today. Atomic and nuclear structure have binding energies of the order of few tens eV and MeV, respectively. Thus, when the temperature of the universe was higher than these values, atoms and nucleons could not have existed as bound objects. Going even further back in time, when the temperature of the photons exceeded the rest mass of the electrons ($m_e \sim 0.5$ MeV), the photons energy was high enough to produce large numbers of electrons and positrons. These particles had the same typical temperature (T), making them ultra-relativistic.

Thus, as a function of temperature, T (or time, t), the universe would be populated by different types of elementary particles. To work out the physical processes at some time t, we need to know the distribution functions $f_A(\vec{x}, \vec{p}, t) \equiv f_A(\vec{p}, t)$ of these particles. Here, A labels the different species of particles. The dependence of f_A on $\vec{x}$ is excluded, because of the homogeneity of the universe.

We can get the distribution function $f_A(\vec{p},t)$ by noting that the different species interact constantly through the various processes, scattering each other and exchanging energy and momentum. As long as the rate of this reactions $\Gamma(t) = n\sigma v$ (n is the density, v is the velocity and σ is the cross section) is much higher than the rate of expansion of the universe, $H(t) = \dot{a}/a$, these interactions produce and maintain thermodynamic equilibrium, with all interacting particles having the same temperature, $T(t)$. We can thus assume that the particles may be treated as *ideal* Bose or Fermi gas, for which the distribution function is given by

$$f_A(\vec{p},t)d^3p = \frac{g_A}{(2\pi)^3}\frac{1}{e^{\frac{E_{\vec{p}}-\mu_A}{T_A(t)}} \pm 1}d^3p \tag{49}$$

where the upper sign (+1) corresponds to Fermions, while the lower sign (-1) to Bosons. Here, g_A is the spin degeneracy factor of the species, $\mu_A(T)$ is the chemical potential, $E_{\vec{p}} = (\vec{p}^2 + m^2)^{1/2}$ and $T_A(t)$ is the temperature of the species at time t.

At any instant in time, the universe also contains black body distribution of photons, with temperature $T_\gamma(t)$. If a particular species of particles is coupled to the photons (namely, $\Gamma_{A\gamma} \gg H$), then these particles will have the same temperature as the photons, $T_A = T_\gamma$. Since this is usually the case, the photon temperature is often referred to as "the temperature of the universe".

As the universe expands, its temperature decreases; furthermore, the number density of particles decreases, and so gradually particles decouple from each other. Once a species A is completely decoupled, all the particles of that species no longer interact (efficiently) with other particles, and they simply move along geodesics. After decoupling, the distribution function freezes; thus, denoting the decoupling time by $t = t_D$, where $a = a_D$, the distribution function at $t > t_D$ is given by

$$f_{\rm dec}(p,t) = f_{\rm equi}\left(p\frac{a(t)}{a_D}\right) \tag{50}$$

where $f_{\rm equi}$ is the distribution function just before decoupling, and we considered the fact that all particles with momentum p at time t must have had momentum $p[a(t)/a_D]$ at t_D.

Given the distribution function, the number density, n, energy density, ρ and pressure, p are given by (omitting the subscript A and the time dependence for clarity)

$$n = \int f(\vec{k})d^3k = \frac{g}{(2\pi^2)}\int_m^\infty \frac{(E^2-m^2)^{1/2}EdE}{e^{\frac{E-\mu}{T}} \pm 1} \tag{51}$$

$$\rho = \int Ef(\vec{k})d^3k = \frac{g}{(2\pi^2)}\int_m^\infty \frac{(E^2-m^2)^{1/2}E^2dE}{e^{\frac{E-\mu}{T}} \pm 1} \tag{52}$$

and

$$p = \int \frac{1}{3}\frac{|\vec{k}|^2}{E} f(\vec{k}) d^3k = \frac{g}{(6\pi^2)} \int_m^\infty \frac{(E^2 - m^2)^{1/2} dE}{e^{\frac{E-\mu}{T}} \pm 1} \tag{53}$$

(I use here $\vec{k}$ to denote momentum, when the pressure is denoted by p). The last equation is obtained by recalling that from the definition of the stress-energy tensor, $T^{\alpha\beta} = k^\alpha dx^\beta/dt = k^\alpha k^\beta/E$ and for an isotropic fluid $p = T^{ii}/3$.

The above expressions for n, ρ and p simplify considerably in some limiting cases. When the particles are highly relativistic ($T \gg m$) and non-degenerate ($T \gg \mu$), we get:

$$\rho \simeq \frac{g}{(2\pi^2)} \int_m^\infty \frac{E^3 dE}{e^{\frac{E}{T}} \pm 1} = \begin{cases} g_B(\pi^2/30)T^4 & \text{(Bosons)} \\ \frac{7}{8} g_F(\pi^2/30)T^4 & \text{(Fermions)} \end{cases} \tag{54}$$

We can therefore express the total energy density contributed by all relativistic species as

$$\rho_{\text{total}} = \sum_{\text{Bosons}} g_i \left(\frac{\pi^2}{30}\right) T_i^4 + \sum_{\text{Fermions}} \frac{7}{8} g_i \left(\frac{\pi^2}{30}\right) T_i^4 = g_{\text{total}} \left(\frac{\pi^2}{30}\right) T^4 \tag{55}$$

where

$$g \equiv g_{\text{total}} = \sum_{\text{Bosons}} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\text{Fermions}} g_i \left(\frac{T_i}{T}\right)^4 . \tag{56}$$

Note that we have considered the possibility that not all the species have the same temperature. If all the species have the same temperature, we have $g_{\text{total}} = g_B + \frac{7}{8} g_F$. Furthermore, the units used here are such that the radiation (Bolzmann's) constant is $a_B = 1$.

The pressure due to the relativistic species is $p \simeq \rho/3 = g(\pi^2/90)T^4$. The number density can be found in a similar way:

$$n \simeq \frac{g}{(2\pi^2)} \int_m^\infty \frac{E^2 dE}{e^{\frac{E}{T}} \pm 1} = \begin{cases} (\zeta(3)/\pi^2) g_B T^3 & \text{(Bosons)} \\ \frac{3}{4}(\zeta(3)/\pi^2) g_F T^3 & \text{(Fermions)} \end{cases} \tag{57}$$

where $\zeta(3) \approx 1.202$ is the Riemann zeta function of order 3. Combining equations (54) and (57), we find that the mean energy per particle, $\langle E \rangle = \rho/n$ is $\approx 2.70T$ for Bosons, and $\approx 3.15T$ for Fermions.

In the opposite limit, $T \ll m$, the exponential in equation (49) is $\gg 1$. In this limit, for both Bosons and Fermions one gets

$$n \simeq \frac{g}{2\pi^2} \int_0^\infty p^2 dp e^{-\frac{m-\mu}{T}} e^{-\frac{p^2}{2mT}} = g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-\frac{m-\mu}{T}} \tag{58}$$

In this limit, $\rho \simeq nm$, and $p = nT \ll \rho$.

Comparison of equations (57) and (58) shows that the number (and energy) density of non-relativistic particles are exponentially damped by a factor $\exp(-m/T)$ with respect to that of relativistic particles. Thus, **at very early times, at $t < t_{eq.}$, in the radiation dominated phase, we may ignore the contribution of non-relativistic particles to the energy density,** ρ.

During the radiation dominate phase, $a(t) \propto t^{1/2}$, and thus

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{1}{4t^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3} g \left(\frac{\pi^2}{30}\right) T^4 \tag{59}$$

We can write this in terms of Planck energy, which provides a natural energy unit, $E_{\rm Pl} = \sqrt{\frac{\hbar c^5}{G}}$, or (using $c = \hbar = 1$) $m_{\rm Pl} = G^{-1/2} = 1.22 \times 10^{19}$ GeV,

$$H(t) = 1.66 g^{1/2} \left(\frac{T^2}{m_{\rm Pl}}\right), \tag{60}$$

and

$$t \simeq 0.3 g^{-1/2} \left(\frac{m_{\rm Pl}}{T^2}\right) \sim 1 \quad \left(\frac{T}{1 \text{ MeV}}\right)^{-2} g^{-1/2} \quad \text{sec.} \tag{61}$$

Remember that the factor g in these expressions counts the degrees of freedom of those particles which are *still relativistic* at the given temperature, T. As the temperature decreases, more and more particles become non relativistic, and g decreases; thus, $g = g(T)$ is a slowly varying function of T. I provide in the appendix the full list of particles in the standard model, together with their masses and degeneracy, from which we calculate which of the particles contribute to the global degeneracy at each temperature.

- For $T \ll$ MeV, the only relativistic particles are the three neutrino species, and the photons. Thus we have 2 degrees of freedoms for the Bosons - photons [polarizations], and 6 degrees of freedom for the Fermions = neutrinos (3 neutrino + anti-neutrino species) ; since $T_\nu = (4/11)^{1/3} T_\gamma$ (see appendix), we have

$$g(T \ll \text{MeV}) = 2 + \frac{7}{8} \times 3 \times 2 \times \left(\frac{4}{11}\right)^{4/3} = 3.36 \tag{62}$$

- For MeV $\lesssim T \lesssim$ 100MeV, the electron and positron add additional relativistic degrees of freedom, (and each one with 2 spins, so total addition of 4 Fermionic degrees of freedom), and one obtains

$$g(\text{MeV} \lesssim T \lesssim 100 \text{ MeV}) = 10.75 \tag{63}$$

- For $T \gg 300$ GeV, all species of the standard model are relativistic, and thus

$$g(T \gtrsim 300\text{ GeV}) = 106.75 \tag{64}$$

The key to understand the thermal history of the universe is the comparison of particle interaction rates to the expansion rate. So long as the interactions necessary for particle distribution functions to adjust to the changing temperature are rapid compared to the expansion rate, the universe will, to a good approximation, evolve through a succession of nearly thermal states with temperature decreasing like a^{-1}.

We can now describe the thermal history of the early universe. Note that we cannot go backward beyond the Planck epoch, $t \sim 10^{-43}$ s, and $T \sim 10^{19}$ GeV (the Planck energy)- the point at which quantum corrections to GR should render it invalid.

At the earliest time, the universe was a plasma of relativistic particles, including quarks, leptons gauge bosons and Higgs bosons. If current ideas are correct, a number of spontaneous symmetry breaking (SSB) phase transition took place during the course of the early history of the universe. They perhaps include the GUT phase transition at $T \sim 10^{16}$ GeV, and the electroweak phase transition at $T \sim 300$ GeV. During these SSB phase transitions, some gauge bosons and matter particles acquire mass through the Higgs mechanism and the full symmetry is broken to lower symmetry. At temperature of about $T \sim 100 - 300$ MeV ($t \sim 10^{-5}$ s), the universe undergoes a transition associated with chiral symmetry breaking and color confinement, after which strongly-interacting particles confine into color singlets combinations - namely, baryons and mesons.

The epoch of nucleosynthesis follows when $t \sim 10^{-2} - 10^{2}$ s ($T \sim 10 - 0.1$ MeV). Neutrons and protons first combine to form D, ^{4}He, ^{3}He, and ^{7}Li nuclei. Quite remarkably, the theory for this is very well developed and agrees very impressively with a variety of observations. At present, nucleosynthesis is the earliest test of standard cosmology.

At time of about 10^{12} s ($T \sim 1$ eV), the energy density in matter becomes equal to that in radiation, and the universe becomes matter dominated. This further marks the beginning of structure formations. Finally, at time of $\sim 10^{13}$ s, ($T \sim 0.1$ eV), ions and electrons combine to form atoms: this is known as **recombination**. When it happens, matter and radiation decouple, ending the long epoch of thermal equilibrium that existed in the early universe. The surface of last scattering for the microwave background radiation (CMB) is the universe itself at decoupling, which occurred at $z_{\rm dec} \approx 1180$ ($T_{\rm dec} \sim 3220^\circ$ K, or 0.28 eV), which occurred when the universe was 378,000 years old.

A. Elementary particles and degrees of freedom

The total number of (effective) degrees of freedom is a function of the temperature of the universe, as well as the particle content of it. In Table 1 I give a list of all the known particles within the standard model of particle physics, their masses and degeneracies.

Species	name	mass		degrees of freedom	Total
Quarks	$u, \bar{u}$	15-3.0 MeV	spin=1/2	g = 2*2*3=12	
	$d, \bar{d}$	3-5 MeV	3 colors		
	$s, \bar{s}$	95 MeV			
	$c, \bar{c}$	1.25 GeV			
	$b, \bar{b}$	4.2 GeV			
	$t, \bar{t}$	175 GeV			
					72
Gluons		8 massless bosons	spin = 1	g=2	16
Leptons	e^-, e^+	511 keV	spin = 1/2	g = 2*2 = 4	
	μ^-, μ^+	105 MeV			
	τ^-, τ^+	1.777 GeV			
					12
	$\nu_e, \bar{\nu}_e$	< 2 eV	spin = 1/2	g = 2[†]	
	$\nu_\mu, \bar{\nu}_\mu$	< 190 keV			
	$\nu_\tau, \bar{\nu}_\tau$	< 18.2 MeV			
					6
Electroweak	W^+, W^-	80.4 GeV	spin = 1	g = 3	
gauge bosons	Z^0	91.2 GeV			
	γ	0		g=2	
					11
Higgs boson	H^0	125 GeV	spin = 0	g=1	1
				$g_F = 72 + 12 + 6 = 90$	
				$g_B = 16 + 11 + 1 = 28$	

Table 1: Degrees of freedom of all standard model particles.
[†] Experimental results show that within the margin of error, all produced and observed neutrinos have left-handed helicities (spins antiparallel to momenta), and all antineutrinos have right-handed helicities. In the massless limit, that means that only one of two possible chiralities is observed for either particle. This is why for each species there are only 2 degrees of freedom, rather than 4 as in the more massive leptons.

At temperatures $T \sim 200$ GeV, all particles are present, relativistic, and in thermal equilibrium, so we find

$$g(T) = 28 + \frac{7}{8} \times 90 = 106.75 \tag{A1}$$

When the temperature drops to $T \sim 1$ GeV, the temperature has dropped below the rest energy of the $t, b, c, \tau, W^+, W^-, Z^0$ and H^0 particles, therefore these are no longer relativistic (and will have annihilated) and we have to take them out of the equation. We are left with

$$g(T) = 18 + \frac{7}{8} \times 50 = 61.75 \tag{A2}$$

total degrees of freedom. When the temperature drops below 100 MeV, the remaining quarks and gluons are locked up in non-relativistic hadrons, and the muons have annihilated. All that's left are photons, electrons, positrons, neutrinos and anti-neutrinos, so that

$$g(T) = 2 + \frac{7}{8} * 10 = 10.75. \tag{A3}$$

So far, all relativistic particles were in thermal equilibrium. However, as the temperature drops to 1 MeV, the neutrinos decouple and move freely, which means their temperature will start to diverge from the photon temperature. At $T < 500$ keV, the electrons and positrons are no longer relativistic, so only the photons and neutrinos remain, and

$$g(T) = 2 + \frac{7}{8} \times 6 \left(\frac{T_\nu}{T}\right), \tag{A4}$$

where T_ν is the neutrino temperature.

Let us now calculate the neutrino temperature. The Second Law of Thermodynamics implies that the entropy density $s(T)$ is given by

$$s(T) = \frac{\rho(T) + p(T)}{T} \tag{A5}$$

where ρ is the energy density and p is the pressure. Using the Fermi-Dirac and Bose-Einstein distributions, we showed (see Equation 54) that for relativistic particles

$$\rho = \begin{cases} g_B(\pi^2/30)a_B T^4 & \text{(Bosons)} \\ \frac{7}{8} g_F(\pi^2/30)a_B T^4 & \text{(Fermions)} \end{cases}$$

and $p = \rho/3$, and so $s(T) = 4\rho(T)/3T$.

Let us now consider the entropy density of the photons and the electrons and positrons at high temperatures, when they are still relativistic:

$$s(T_{\rm high}) = \frac{4}{3} a_B T_{\rm high}^3 \left(2 + \frac{7}{8} * 4\right) = \frac{4}{3} a_B T_{\rm high}^3 \left(\frac{11}{4}\right) \tag{A6}$$

where 2 degrees of freedom are from the photons (Bosons), and 4 are from the Fermions - electrons and positrons (2 each, for the spin).

At low temperatures, the electrons and positrons become non-relativistic, most annihilate and the remaining particles have negligible contribution to the entropy, therefore

$$s(T_{\rm low}) = \frac{4}{3} a_B T_{\rm low}^3 \tag{A7}$$

Thermal equilibrium implies that the entropy in a comoving volume remains constant:

$$s(T)a^3 = \text{const}$$

After decoupling, the temperature of the neutrinos drops as $T_\nu \propto a^{-1}$. Combining these results, we find

$$\left(\frac{T_{\rm low}}{T_{\nu,\rm low}}\right)^3 = \frac{11}{4}\left(\frac{T_{\rm high}}{T_{\nu,\rm high}}\right)^3 \tag{A8}$$

At high temperatures, the neutrinos are still in thermal equilibrium with the photons, i.e. $T_{\nu,\rm high} = T_{\rm high}$. Thus, after neutrino decoupling we finally obtain

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T \tag{A9}$$

at low temperatures. Therefore,

$$g(T) = 2 + \frac{7}{8} * 6 * \left(\frac{4}{11}\right)^{1/3} = 3.36. \tag{A10}$$

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