From the big bang to large scale structure

Asaf Pe'er[1]

March 15, 2017

This part of the course is based on Refs. [1] - [3].

1. A brief overview on basic cosmology

As this part was covered by Bryan, I will only go briefly over the key results. It is the student's responsibility to ensure that all that is written here is clear to you. The material in this section is based on the "Cosmology" part in the GR course (PY4112), which is needed to get full understanding of the FRW universe. As it is not defined as a pre-requisite, if you didn't learn GR, simply accept FRW equations as given.

The basic principle which is in the heart of the theory of general relativity (GR) is Einstein's radical idea that **space itself is not fixed but flexible**. Note this is conceptually different than anything else we are used to. We are used to think as space as "fixed", like a blackboard on top of which objects such as particles move as they interact with each other. As opposed to that, Einstein showed that the space is flexible, and can shrink or expand - very much like a rubber band. The "thing" that shrinks space (space-time) is energy, or mass (since $E = mc^2$). Mathematically, the relation between the curvature of space-time and the energy content is described by *Einstein's equation*,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}. \tag{1}$$

Here, $G_{\mu\nu}$ is Einstein's tensor, $T_{\mu\nu}$ is the stress-energy tensor, G is Newton's gravitational constant, Λ is a constant (known as the cosmological constant), and $g_{\mu\nu}$ is the metric tensor (if you took GR, the meaning of all these should be perfectly clear to you, and if not, it is OK- I will not continue in this direction).

It can be shown that a solution of Einstein's equation for a universe that is *homogeneous* and *isotropic* namely its spatial term is **maximally symmetric** is given by the Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] \ . \tag{2}$$

[1]Physics Dep., University College Cork

The curvature constant k can get three values, $k = -1, 0, +1$ describing an **open, flat and closed** universes, respectively. The term "metric" is central to GR, as well as to branch of mathematics known as differential geometry. Very loosely, it describes the distance between two points. Note that for **flat** space, the metric takes the form

$$ds^2 = -dt^2 + a^2(t)\left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] = -dt^2 + a^2(t)\left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right], \tag{3}$$

which is indeed Euclidean (flat); but the spatial distance between any two object changes with time, **even if the objects are fixed in space**.

The evolution of the **scale factor** $a(t)$ is obtained by using this metric in Einstein's equation. The results are two equations,

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)\ , \tag{4}$$

and

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\ , \tag{5}$$

which are known together as **Friedmann Equations**. Here, ρ is the *energy density* and p is the pressure of the content of the universe (matter [particles and dark matter] and radiation). These equations describe the evolution of the scale factor, hence of the universe as a whole.

1.1. Hubble's law and the big bang

As was proven by the astronomer **Edwin Hubble** in 1929, all galaxies recede away from us, at velocities that are proportional to the distance from us.

Mathematically, one can define **Hubble's constant**, H_0 and the **deceleration parameter** q_0 as

$$H_0 \equiv \left(\frac{\dot{a}}{a}\right)_{t=t_0} \quad ; \qquad q_0 \equiv -\left(\frac{a\ddot{a}}{\dot{a}^2}\right)_{t=t_0} . \tag{6}$$

where t_0 is today. (note that Hubble's constant is only constant today - it changes with time, since the scale factor $a(t)$ change in time ! and the same is true for the deceleration parameter). The latest measured value I found in Wikipedia is $H_0 = 71.9^{+2.4}_{-3.0}$ (km/s)/Mpc measured in Nov. 2016, where Mpc stand for Mega-parsec (and 1 pc $\approx 3 \times 10^{18}$ cm).

It is often convenient to write Hubbles constant as

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}, \tag{7}$$

making $h \approx 0.7$ a dimensionless constant.

Hubble's law can be written as

$$d_L = H_0^{-1}\left[z + \frac{1}{2}(1-q_0)z^2 + \ldots\right] . \tag{8}$$

Here, d_L is the **luminosity distance**, which is defined by

$$d_L^2 = \frac{L}{4\pi F} , \tag{9}$$

where L is the absolute luminosity of a given source and F is the flux measured by the observer (the energy per unit time per unit area of some detector); note that as the universe expands, one need to be very precise about what method is used to measure a distance to an object.

The redshift, z is another measurable quantity, which is defined by

$$\begin{aligned} z &= \tfrac{\lambda_0-\lambda_1}{\lambda_1} \\ &= \tfrac{a_0}{a_1} - 1 . \end{aligned} \tag{10}$$

The subscript X_0 represent "today" (t_0), and X_1 represents a quantity measured at some earlier time $t_1 < t_0$. Thus, a_0 is the value of the scale factor today, while $a_1 \equiv a(t = t_1)$. Similarly, λ_0 is the wavelength of a photon today, while λ_1 is the wavelength of the same photon at earlier time (the wavelength of a photon changed since the universe expands). This is a quantity that is relatively easy to measure, by looking at known emission and absorption lines of different elements that are common to all galaxies.

Hubble's discovery, that $H_0 > 0$ (which is immediate from Hubble's law taken to first order, $z = d_L H_0$) implies that $\dot{a} > 0$. Using Friedmann equations given above, it is easy to show that for universes filled with fluids of positive energy ($\rho > 0$), nonnegative pressure ($p \geq 0$) and no cosmological constant ($\Lambda = 0$), one finds $\ddot{a} < 0$.

We can therefore trace the evolution of the universe backward in time, and we necessarily reach a singularity, namely $a = 0$. Notice that if $\ddot{a}$ were exactly zero, $a(t)$ would be a straight line, and the age of the universe would be H_0^{-1}. Since $\ddot{a}$ is actually negative, the universe must be somewhat younger than that. This is demonstrated in Figure 1.

This singularity at $a = 0$ is the **Big Bang**. It represents the creation of the universe from a singular state, **not** explosion of matter into a pre-existing spacetime. Note that the universe can still be spatially infinite and flat (Equation 3); simply, all coordinated would shrink into zero. An interesting analogy is with a copying machine, copying an infinitely large sheet of paper while using the zoom out option. Every copy, the information gets denser and denser, but if we began with an infinitely big paper, it is still infinitely big. Asymptotically, the density gets to infinity, and the laws of physics break.

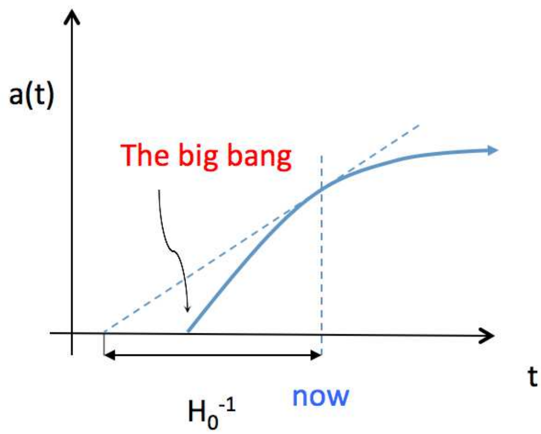

Fig. 1.— In a universe with no cosmological constant, we know that it is expanding ($\dot{a} > 0$) and decelerating ($\ddot{a} < 0$). Thus, there must have been a point in the past when $a = 0$. This is the Big Bang.

1.2. Evolution of the scale factor $a(t)$; brief history of the universe.

Friedmann's Equation (Equations 4 and 5) can be solved to obtain the evolution of $a(t)$ in the various scenarios. Since the universe is composed of matter, radiation and vacuum energy, the evolution depends on answering two questions: (i) whether the universe is flat, open or closed ($k = 0, -1, +1$); and (ii) what is the dominant energy content of the universe.

In the Cosmology part in GR, we showed that for all three ingredients (matter, radiation and vacuum energy), we can write an equation of state in the form $p = \omega\rho$, from which the conservation of energy becomes

$$p = \omega\rho \Rightarrow \rho \propto a^{-3(1+\omega)} \Rightarrow \begin{cases} \rho_m \propto a^{-3} & [\omega = 0] \\ \rho_r \propto a^{-4} & [\omega = 1/3] \\ \rho_\Lambda \propto a^0 & [\omega = -1] \end{cases} \tag{11}$$

Where ρ_r is the energy density in both radiation and relativistic matter, ρ_m is the energy density in non-relativistic matter and ρ_Λ is the vacuum energy density. Thus,

$$\rho_m(t) = \rho_m(t_0)\left(\frac{a_0}{a}\right)^3 = \rho_c\Omega_m(1+z)^3 \tag{12}$$

where $\Omega_m \equiv \rho_{m,0}/\rho_c$; $\rho_c = \rho_{crit,0} = 3H_0^2/8\pi G(= 8.62 \times 10^{-30}\ \ \mathrm{g\,cm^{-3}})$ is the critical density; and the subscript $_0$ represents present time values.

Similarly,

$$\rho_r(t) = \rho_r(t_0)\left(\frac{a_0}{a}\right)^4 = \rho_c\Omega_r(1+z)^4, \tag{13}$$

and

$$\rho_\Lambda(t) = \rho_\Lambda(t_0). \tag{14}$$

Observations suggest that at present epoch

$$\Omega_{\rm total} \equiv \Omega \simeq 1; \quad \Omega_m \sim 0.3; \quad \Omega_r \simeq 8.2 \times 10^{-5} \tag{15}$$

Thus, at present, matter dominates over radiation (and vacuum over both). But when looking into equations 12, 13 and 14, clearly when looking at the past (z increases), radiation energy density grows faster than matter (and vacuum) energy densities as we go to earlier phases of the universe. At some time, $t = t_{eq}$ in the past (corresponding to a value $a = a_{eq}$ and redshift $z = z_{eq}$) the radiation and matter have had equal energy densities. From Equations 12, 13 and 15 we get

$$(1+z_{eq}) = \frac{a_0}{a_{eq}} = \frac{\Omega_m}{\Omega_r} \simeq 3.6 \times 10^3 \tag{16}$$

Since the temperature of the radiation grows as a^{-1} (recall Stefan-Boltzmann's law, $T^4 \propto \rho_r \propto (1+z)^4$), the temperature of the universe at this epoch was

$$T_{eq} = T_0(1+z_{eq}) = 2.7 \times 3.6 \times 10^3 \simeq 1.0 \times 10^{4\circ}\text{K} = 0.9 \text{ eV} \tag{17}$$

where $T_0 = 2.7^\circ$ K is the (current) temperature of the cosmic microwave background (CMB) radiation.

We can now solve Friedmann Equation (Equation 4) for various geometries and contents of the universe. Let us focus on the flat universe ($k = 0$). For matter dominated universe, we find

$$a(t) = \left(\frac{9}{4}\frac{8\pi G}{3}\Omega_m\rho_c a_0^3\right)^{1/3} t^{2/3}, \tag{18}$$

while for radiation-dominated flat universe

$$a(t) = \left(4 \times \frac{8\pi G}{3}\Omega_r\rho_c a_0^4\right)^{1/4} t^{1/2}. \tag{19}$$

Clearly, at very early times, $t \ll t_{eq}$, the energy density of the universe is dominated by radiation, while at later times, $t \gg t_{eq}$ it is dominated by matter. Given that we can consider present day epoch as being matter-dominated, the Equilibrium time, t_{eq} is easily obtained using Equations 16 and 18.

As the universe expanded, it cooled. At time of about 10^{12} s ($T \sim 1$ eV), the energy density in matter becomes equal to that in radiation, and the universe becomes matter dominated. This further marks the beginning of structure formations. Finally, at time of $\sim 10^{13}$ s, ($T \sim 0.1$ eV), ions and electrons combine to form atoms: this is known as **recombination**. When it happens, matter and radiation decouple, ending the long epoch of thermal equilibrium that existed in the early universe. The surface of last scattering for the microwave background radiation (CMB) is the universe itself at decoupling, which occurred at $z_{\rm dec} \approx 1180$ ($T_{\rm dec} \sim 3220^\circ$ K, or 0.28 eV), which occurred when the universe was 378,000 years old.

2. Inhomogeneous universe: linear perturbation theory

The discussion on cosmology so far assumed that the universe is homogeneous. However, clearly the universe we see today is not homogeneous, but rather contain structures on many different scales: from the solar system (scale of, say, 1 A.U.$= 1.5 \times 10^{13}$ cm), galaxies (typical scale of 10 kpc, where 1kpc $= 3.15 \times 10^{18}$ cm), galaxy clusters (typical scale of ~ 1 Mpc, and super-clusters (~ 30 Mpc).

Inhomogeneities on small scales, $\ll 10$ Mpc are highly non-linear. This means that the fractional density fluctuations on this scale, $\delta \equiv \delta\rho/\rho$, where ρ is the average density, are large: $\delta \gg 1$. On scale of ~ 10 Mpc, $\delta \approx 1$, while on much larger scales, $\delta \ll 1$.

As opposed to that, when looking at the cosmic microwave background (CMB), the radiation is highly isotropic. Variations in the CMB does not exceed $\approx 10^{-5}$. The CMB radiation decoupled from matter at $z_{\rm dec} = 1180$, and therefore the CMB anisotropy reflects the level of anisotropy at that early epoch. The smallness of the CMB fluctuations imply that the universe has been nearly homogeneous at $z = z_{\rm dec}$. We therefore believe that the structures we see today are the results of evolution of these small perturbations, which were generated at an early epoch of the universe evolution.

Furthermore, these observations suggest that during most of cosmological time, the evolution of perturbaton was **linear**. Only relatively recently the evolution of perturbations became non-linear on scales smaller than ~ 10 Mpc. This is in agreement with the cosmological principle.

As in standard cosmology we treat both radiation as well as matter (regular and dark matter) as **ideal fluid**, the description of perturbations is similar to the description of perturbation in fluids. In particular, we are interested in studying under what conditions perturbations grow, and when they are suppressed.

Basically, one can distinguish between two types of perturbations. The first is **isentropic** perturbations, which are pure density perturbations. While the density of the fluid is perturbed, its *entropy* is conseved; hence it is appropriate to call these *adiabatic* perturbations. The second type of perturbations involve perturbation of the entropy of the fluid. Such perturbations do not represent perturbation of the metric, but rather of the equation of state (ratio of baryons per photons). These are also known as **isocurvature** perturbations (sometimes these are also called **isothermal** perturbations, as at $t \ll t_{\rm eq.}$ in such perturbations, the temperaute is approximately conserved. In general, any perturbation can be written as a linear combination of both isentropic and isocurvature perturbations.

Perturbations grow due to the effect of gravity. Gravity slows the expansion of the universe. As a result, a slightly over-dense region will suffer slower than average deceleration, leading to slower than average expansion. This will lead to an increase of the over-density with time (positive feedback). However, as we will discuss below, there are several effects that will reduce this positive feedback.

Here we will discuss the evolution of the perturbation in the linear regime. Since the initial perturbations were small, $\delta \ll 1$, much of their evolution may be described by a linear perturbation theory. For large scale perturbation, $\gtrsim 10$ Mpc, this is true all the way until present epoch, while at smaller scale at a certain point the perturbation become non-linear.

3. Hubble radius

As the universe expands, quantities such as density, etc, evolve. A characteristic time scale for this evolution is $(\dot{a}/a)^{-1} = H(t)^{-1}$. This naturally introduces a characteristic length, known as the **Hubble radius**,

$$\lambda_H \equiv \frac{c}{H} = c\frac{a}{\dot{a}} \tag{20}$$

Since no signal travels faster than the speed of light, the Hubble's radius has an important physical significance. **It represents the largest length scale that is causally connected**, i.e., across which information can be communicated, during the time over which significant expansion (or perturbation growth) takes place.

When dealing with causally-connected regions, due to the expansion of the universe we have to be careful. We normally talk about the **proper distance**, $L_{\rm prop}$, which is the distance between two points at some time t in the past. However, we like to work in "comoving units": due to the expansion of the universe, the distance increases linearly with a. Thus, two points separated by a proper length $L_{\rm prop}$ at some time t are separated at present time ($t = t_0$) by $L_{\rm prop} \times a_0/a(t) = L_{\rm prop}/a(t)$. Here and below, we take $a(t = t_0) = 1$.

This means that a perturbation on scale $\lambda_{\rm prop}$ at time t corresponds at present ($t = t_0$) to a perturbation on scale $\lambda_{\rm co.} = \lambda_{\rm prop}/a(t)$. In what follows, I may simply drop the subscript "co.", and express all length scales in comoving units.

As we saw above (Equations 18, 19), the scale factor $a(t) \propto t^n$, where the power law index n depends on whether the universe is matter dominated ($n = 2/3$) or radiation dominated ($n = 1/2$). At time $t = t_{\rm eq.}$ in the past, the matter and energy densities were equal; the scale factor at that time is $a = a_{\bf eq.}$. Its value was calculated in equation 16, and can be written as $a_{\rm eq.}^{-1} = 2.4 \times 10^4 \Omega_m h^2$.

During the matter dominated era, $a_{\rm eq.} < a < 1$, the scale factor evolves according to Equation 18, $\dot{a}^2 = \Omega_m H_0^2 a_0^3 a(t)^{-1}$. Earlier, during radiation dominated era, $a < a_{\rm eq.}$ Equation 19 gives $\dot{a}^2 = \Omega_r H_0^2 a_0^4 a(t)^{-2}$. Thus, at equilibrium, the (comoving) Hubble radius is

$$\lambda_{\rm eq.} = \frac{c}{\dot{a}_{\rm eq.}} \simeq 130\ (\Omega_{m,0.3} h_{70}^2)^{-1}\ {\rm Mpc}. \tag{21}$$

Before and after equilibrium, Hubble radius is

$$\lambda_H \equiv \frac{c}{\dot{a}} = \lambda_{\rm eq.} \times \left\{ \begin{array}{ll} (a/a_{\rm eq.})^1 & a \ll a_{\rm eq.}; \\ (a/a_{\rm eq.})^{1/2} & a \gg a_{\rm eq.}. \end{array} \right. \tag{22}$$

The baryon mass enclosed in a sphere of radius λ is

$$M_\lambda = \frac{4\pi}{3} \lambda^3 \Omega_b \rho_c = 7.5 \times 10^9 \Omega_{b,-2} h_{70}^2 \lambda_{\rm Mpc}^3\ M_\odot. \tag{23}$$

Here, we took $\Omega_b = 10^{-2}\Omega_{b,-2}$ is the baryon energy density today (measured in units of ρ_c), and $M_\odot$ is the solar mass. Note that here and below, we use a different notation to the baryon energy density Ω_b and matter energy density, Ω_m to to allow for the presence of non-baryon matter (dark matter). We will further elaborate on that below. Since this is about the mass in a galaxy ($\sim 10^{10} M_\odot$), we can deduce that a galaxy was formed out of fluctuations on scale of ~ 1 Mpc.

4. The Jeans scale: baryonic matter

The physical situation is as follows. At scales $\lambda < \lambda_H$, gravity acts to increase the density fluctuations. However, at the same time, plasma pressure acts to suppress them. When a region gets denser, its expansion slows down, and therefore the rate of pressure decrease in this region also slows down relative to its environment. As a result, this region becomes over-pressured, and this over-pressure will act to accelerate the over dense region's expansion.

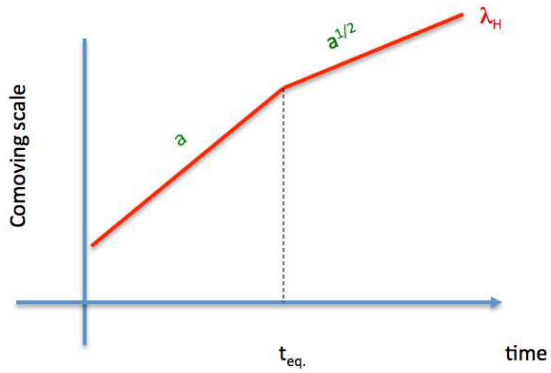

Fig. 2.— Above the Hubble radius, regions cannot be causally connected.

We want now to estimate the scale at which these pressure effects are important. For that, we recall that we treat the universe as an ideal fluid. In such a fluid, pressure perturbations propagate at the speed of sound, c_s. Before recombination, we have a coupled baryon-photon fluid, and the speed of sound can be written as

$$c_s = \frac{c}{\sqrt{3}}\left[\frac{3}{4}\frac{\rho_b(t)}{\rho_r(t)} + 1\right]^{-1/2}. \tag{24}$$

Since $\rho_b \propto a^{-3}$ and $\rho_r \propto a^{-4}$, we need to discriminate between two epochs.

- At $t < t_{\rm eq.}$, $\rho_b(t) < \rho_r(t)$ and $c_s = c/\sqrt{3} \propto a^0$.
- At $t > t_{\rm eq.}$, $\rho_b(t)/\rho_r(t) \propto a$ and $\rho_b(t)/\rho_r(t) > 1$, and so $c_s \propto a^{-1/2}$.

We thus find that over-pressure will prevent the growth of perturbations on a scale for which the sound crossing time, $\lambda_{\rm prop.}/c_s$ is shorter than the time scale for the growth of the perturbation, which is $a/\dot{a}$. In other words, $\lambda_{\rm prop} < c_s a/\dot{a}$, correspsonding to comoving scale $\lambda_{\rm co.} < c_s/\dot{a}$. This enables to define the **Jeans scale**,

$$\lambda_J \equiv \frac{c_s}{\dot{a}} = \frac{c_s}{c}\lambda_H. \tag{25}$$

Note that some textbooks use somewhat different definition of the Jeans scale, as

$$\lambda_J = \sqrt{\pi}\frac{c_s}{\sqrt{G\rho}} \tag{26}$$

This follows from comparing the free fall time due to gravity, $t_{ff} \sim (G\rho)^{-1/2}$ (which you can derive from Kepler's laws !), with the time it takes the pressure to re-adjust the density, which is λ_J/c_s. The two definitions are similar, up to a factor of order unity.

For $\lambda > \lambda_J$, pressure effects may be neglected, and perturbations will grow; on the other hand, for $\lambda < \lambda_J$, pressure prevents the growth of over perturbations, and rather leads to density oscillations (acoustic waves). This oscillatory behavior can be understood in terms of energy conservation. When the plasma in some region of the universe is compressed, its internal energy is increased. When it expands, the excess of internal energy is converted to excess kinetic energy, leading to faster than average expansion. This, in turn, leads to compression of the surrounding plasma, converting the excess kinetic energy to excess internal energy of the surrounding plasma. The process then repeats itself.

The evolution of the Jeans scale is readily obtained:

- For $a < a_{\rm eq.}$, the plasma is radiation dominated, and $c_s = c/\sqrt{3}$. Thus, $\lambda_J = \lambda_H/\sqrt{3}$.
- At $a > a_{\rm eq.}$, we had $c_s \propto a^{-1/2}$. Since in this regime $\lambda_H \propto a^{1/2}$, we have $\lambda_J \propto a^0$.

  Using the Jeans length, we can define the Jeans mass as

$$M_J \equiv \frac{4\pi}{3}\rho\left(\frac{\lambda_J}{2}\right)^3. \tag{27}$$

  This is the mass enclosed in a sphere of radius $\lambda_J/2$. At $a = a_{\rm eq.}$ we therefore find that

$$M_J = \frac{4\pi}{3}\rho_b\left(\frac{\lambda_J}{2}\right)^3 \simeq 2\times 10^{16}(\Omega_{b,0}h_{70}^2)^{-2}M_\odot \tag{28}$$

  This is the approximate mass of a supercluster of galaxies. Thus, perturbations on scales smaller than that of a supercluster cannot grow before recombination.

- On the other hand, at recombination baryons decouple the photons. As a result, the baryons no longer feel the photon pressure, and the pressure term is dominated by the thermal motion of the baryons themself. At this stage, the baryons are non-relativistic. Therefore,

$$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_S = \left(\frac{5}{3}\right)\frac{k_B T}{m_p} \tag{29}$$

  Here, T is the baryon temperature. We thus find that $c_s \propto T^{1/2}$. Furthermore, note the very sharp drop in the value of c_s that follows the decoupling. As photons and particles are decoupled, the radiative pressure does not play any further role, and is replaced by baryon pressure, which is $p_b \ll p_r$ due to the fact that the number of photons outnumber that of the baryons by $n_r : n_b \sim 10^8 : 1$.

Following decoupling, $T = 0.35(a/a_{\rm dec.})^{-2}$ eV. The value of T is derived from Equation 16 by recalling that $z_{\rm dec.} \approx 1100$. The scaling of T with the scale factor a can easily be understood by noting that for non-relativistic particles, their velocity decays as $v \propto a^{-1}$ due to the expansion, and $T \propto \langle E \rangle \propto v^2$, where $\langle E \rangle$ is the average energy.

We therefore find that

$$\frac{c_s}{c} = 2.5 \times 10^{-5} (a/a_{dec.})^{-1}, \tag{30}$$

and

$$\lambda_J = 2 \times 10^{-5} \lambda_{\rm eq.} (a/a_{dec.})^{-1/2}. \tag{31}$$

Thus, immediately after recombination, $M_J = 1.5 \times 10^5 (\Omega_{b,0} h^2)^{-1/2} M_\odot$. This is about the scale of a globular cluster. Comparing to equation 28, we see a decrease by 11 orders of magnitude at recombination !. The decoupling of the photons cause huge drop in Jeans mass.

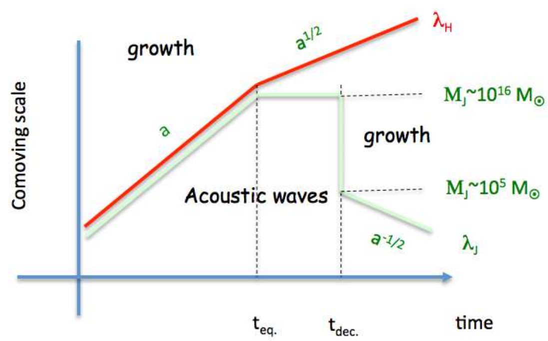

Fig. 3.— Scales for growth of **baryonic** perturbations

5. Silk damping

Before decoupling, photons and baryons are strongly coupled. However, the coupling is imperfect; photons have mean free path $\langle l \rangle = (\sigma_T n_e)^{-1}$ with is greater than 0. Here, $\sigma_T \approx 6.65 \times 10^{-25}$ cm^2 is Thomson cross section. In other words, photons diffuse from over-dense region. This diffusion cause damping of perturbation in the photon distribution, hence damping of the acoustic oscillations. This is known as **Silk damping**, after Joe Silk.

The Silk damping scale, λ_d is the typical distance photons can diffuse during Hubble time. It can be calculate it as follows. During time t, a photon takes on the average $N = ct/\langle l \rangle$ steps. From kinetic theory, we have

$$\lambda_d = \sqrt{\frac{N}{3}}\langle l \rangle = \sqrt{\frac{ct}{3\langle l \rangle}}\langle l \rangle = \left(\frac{ct}{3\sigma_T n_e}\right)^{1/2}. \tag{32}$$

The available time for photons diffusion is $t = a/\dot{a}$. Thus, the comoving scale for Silk damping (the scale below which perturbations will be suppressed) is

$$\lambda < \lambda'_D(a) = \frac{1}{a}\left(\frac{ca}{3\dot{a}\sigma_T n_e}\right)^{1/2} = \left(\frac{c}{3a\dot{a}\sigma_T n_e}\right)^{1/2}. \tag{33}$$

The evolution of the number density is given by $n_e = \Omega_b(\rho_c/m_p)a^{-3}$. In the radiation dominated era, $a \ll a_{\rm eq.}$, we have $\dot{a}^2 = \Omega_r H_0^2 a_0^4 a(t)^{-2} = \Omega_m H_0^2 a_0^3 a_{\rm eq.}/a(t)^2$, where use was made in Equation 16. In the radiation dominated era, we thus find that the diffusion scale grows with time,

$$\lambda'_D(a) \approx \left(\frac{m_p c}{3\sigma_T \Omega_b \rho_c \Omega_m^{1/2} a_{\rm eq.}^{1/2} H_0 a_0^{3/2}}\right)^{1/2} a^{3/2}. \tag{34}$$

At equilibration time, it is equal to

$$\lambda_{D,\rm eq.} = \lambda'_D(a = a_{\rm eq.}) \approx 4(\Omega_{m,0.3}/\Omega_{b,0.02})^{1/2}(\Omega_{m,0.3}h_{70}^2)^{-2}\ {\rm Mpc}. \tag{35}$$

At later times, $t > t_{\rm eq.}$, $\dot{a}^2 \propto a^{-1}$. As a result, $\lambda'_D(a) \propto a^{5/4}$. Thus, at recombination, $\lambda'_D(a_{\rm dec}) \approx 17$ Mpc. The mass enclosed is

$$M_D = \frac{4\pi}{3}\rho\left(\frac{\lambda'_D}{2}\right)^3 \sim 10^{13}\ \left(\frac{\Omega_m}{\Omega_b}\right)^{3/2}\left(\Omega_m h^2\right)^{-5/4}\ M_\odot \tag{36}$$

Silk damping prevents the survival of any baryonic perturbations occuring on scales smaller than λ_D. It is clearly observed when looking at fluctuations of the CMB.

We thus find that if matter is purely baryonic and perturbations are isentropic (adiabatic), structure formation proceeds **top-down**, by fragmentation of perturbations larger than Silk damping scale at recombination, $M_d \sim 10^{13} M_\odot$.

In an alternative model (by Peebles), perturbations are isothermal. In this model, the sound speed is much lower, and as a result so is the Jeans mass. In this model, there are no radiation perturbations, there is no Silk damping, and all perturbations with $M > M_J \sim 10^6 M_\odot$ survive. In this model, thus, structure formation is **Hierarchical** (bottom-up).

Both models, though, are inconsistent with the observed fluctuations in the CMB...

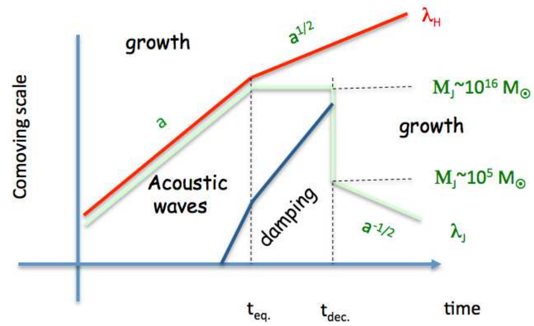

Fig. 4.— Silk (diffusion) damping erases perturbations on scales $\lambda < \lambda_D$.

6. Evolution of the perturbation

6.1. Large scales: $\lambda \gg \lambda_H$

Note: I give a somewhat "sketchy" derivation, as I don't want to get into too many technicalities here.

Using the definition of the Hubble's constant (eq. 6), and $\Omega = \rho/\rho_c$, we can write Friedmann Equation (4) as

$$H^2(\Omega - 1) = \frac{k}{a^2} \tag{37}$$

Let us consider a spherical region, of radius $\lambda > \lambda_H$, containing a matter at density $\rho_1 = \rho_0 + \delta\rho$, which is embedded in $k = 0$ universe of density ρ_0. Due to spherical symmetry (more accurately: Birkhoff's theorem), the inner region is not affected by the matter outside. As a result, the inner region, being over-denee ($\delta\rho > 0$) evolves like a $k = +1$ universe. We can therefore write, for the two regions:

$$H_1^2 + \frac{1}{a_1^2} = \frac{8\pi G}{3}\rho_1 \ ; \qquad H_0^2 = \frac{8\pi G}{3}\rho_0, \tag{38}$$

where $H_1 = (\dot{a}_1/a_1)$ and $H_0 = (\dot{a}_0/a_0)$. We compare the perturbed universe with the background universe when their expansion rates are equal; i.e., at time t when $H_1 = H_0$. At that time, we get

$$\frac{8\pi G}{3}(\rho_1 - \rho_0) = \frac{1}{a_1^2}, \tag{39}$$

or

$$\frac{\rho_1 = \rho_0}{\rho_0} = \frac{\delta\rho}{\rho_0} = \frac{3}{8\pi G(\rho_0 a_1^2)}. \tag{40}$$

For small $\delta\rho/\rho_0$, although in general $a_1 \neq a_0$ they are close, and we can write $a_1 \approx a_0$. Since in radiation dominated era ($t < t_{\rm eq.}$) we have $\rho_0 \propto a^{-4}$ while in matter dominated era ($t > t_{\rm eq.}$), $\rho_0 \propto a^{-3}$, we get

$$\left(\frac{\delta\rho}{\rho}\right) \propto \begin{cases} a^2 & (t < t_{\rm eq.}) \\ a & (t > t_{\rm eq.}) \end{cases} \tag{41}$$

We thus find that the amplitude of modes with $\lambda > \lambda_H$ always grows.

6.2. Inside the Hubble radius

Let us consider what happens to the modes inside the Hubble radius. As we already argued, if the mode enters the Jeans radius, it will stop growing, and instead oscillates. Inside λ_D it will inevitably decay. However, if the mode has length $\lambda > \lambda_J$, the arguments used in deriving equation 41 still hold; as a result, the mode will continue to grow with amplitude $\delta\rho/\rho \propto a$.

This hand waving argument is only correct for a flat universe with no dark matter, namely $\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$. This is known as "Einstein-de Sitter (EdS) universe. As we know today, the universe is more complicated than this, as $\Omega_\Lambda \simeq 0.7$ and $\Omega_m \simeq 0.3$. For such a universe the growth rate is a more complicated function of time. A good approximation is derived in MVW (section 4.1.6),

$$\delta \propto D(z) \propto \frac{g(z)}{1+z} \tag{42}$$

where

$$g(z) \simeq \frac{5}{2}\Omega_m(z)\left\{\Omega_m^{4/7}(z) - \Omega_\Lambda(z) + \left[1 + \frac{\Omega_m(z)}{2}\right]\left[1 + \frac{\Omega_\Lambda(z)}{70}\right]\right\}^{-1}. \tag{43}$$

The important point is that due to the expansion of the universe, this growth rate is some **power law** in time, rather than exponential.

Calculation of the growth rate provides an independent method to study the content of the universe. The amplitude of perturbations today is inferred from galaxy surveys, while the amplitude of perturbations at decoupling is inferred from studying the anisotropy in the CMB radiation. Thus, by knowing the factor by which the perturbation amplitude has grown from decoupling till today, $\delta(t_0)/\delta_1(t_{\rm dec.})$ one can constrain the cosmological parameters. For EdS universe, we already derived $\delta(t_0)/\delta_1(t_{\rm dec.}) = 1/a_{\rm dec.}$.

7. Dark matter

By definition, "dark matter" is matter that we cannot observe through electromagnetic interaction (emission and absorption of photons). The existence of which is therefore inferred from gravitational effects only. Based on observations of large scale objects (clusters of galaxies), it is inferred that the density of dark matter in the universe exceeds that of normal (luminous) matter, by a factor of 6:1 or so.

Dark matter is not necessarily non-baryonic. It may be ordinary, baryonic matter, which is in the form that is difficult to detect electromagnetically, such as rarefied hot gas with little emission and absorption, low mass stars which can be very dim, etc. Here, though, we treat it as non-baryonic, and consider its effect on the evolution of the perturbations.

Because dark matter has little interaction with photons, it decouples the rest of the plasma (matter and radiation) at some time $t_{\rm dec.,DM}$ before $t_{\rm dec.}$. We distinguish between two possibilities. The first is that during its decoupling from the rest of matter and radiation, the dark matter particles are relativistic. This is called **hot dark matter**. In the second option, the dark matter particles are already non-relativistic at decoupling. This is **cold dark matter**.

7.1. Evolution of perturbation

During radiation dominated era, the evolution of perturbations is governed by radiation, and there is no difference between the regular and dark matter. At $a > a_{\rm eq.}$ the evolution of perturbations is governed by matter. Our previous discussion on the evolution of perturbations on scale $\lambda > \lambda_J \simeq \lambda_{\rm eq.}$ holds for both ordinary and dark matter. As this evolution is governed by gravity only, it is the same for both regular and dark matter.

However, there is a difference between the evolution of perturbations at $a > a_{\rm eq.}$ on smaller scales, $\lambda < \lambda_J \simeq \lambda_{\rm eq.}$. Recall that at $a < a_{dec.}$, the Jeans scale is determined by radiation pressure. This pressure is what prevents perturbations in the baryonic matter component from growing on small scales, $\lambda < \lambda_J \simeq \lambda_{\rm eq.}$. On this scale, the baryon density perturbations oscillate during $a_{\rm eq.} < a < a_{dec.}$.

However, dark matter particles are not coupled to the radiation, and therefore they do not "feel" the radiation pressure. If the density of dark matter is much greater than that of baryons, $\Omega_m \gg \Omega_b$, the baryonic density oscillations have only little effect on the distribution of dark matter. In this case, perturbations in the dark matter component on all scales, including $\lambda < \lambda_{\rm eq.}$ will continue to grow during $a_{\rm eq.} < a < a_{dec.}$, as $\delta \propto a$.

This grow of dark matter perturbation during $a_{\rm eq.} < a < a_{dec.}$ affects baryon perturbations as well. After decoupling, the baryon Jeans scale drops by many orders of magnitude, as radiation pressure no longer supports the baryons. They therefore fall into the gravitational potential wells created by the dark matter. Thus, in the presence of dark matter, baryon perturbations on scale $\lambda \lesssim \lambda_{\rm eq.}$ are amplified by a factor $a_{dec.}/a_{\rm eq.} = 20\Omega_m h_{75}^2$ beyond their amplification in the absence of dark matter.

Although dark matter particles do not "feel" the radiation pressure, dark matter perturbations on small scales are suppressed due to a different reason. This suppression will obviously only be important for $\lambda \ll \lambda_H$, namely at $a > a_{\rm eq.}$. In this regime, we can derive the the gravitational evolution of the dark matter perturbations as follows. (We ignore the expansion of a homogeneous universe, which will not change our conclusion).

Since dark matter is **collisionless**, we cannot use a fluid description in determines its evolution. Instead, we use the **collisionless Boltzmann equation**. Let $f(\vec{x},\vec{v},t)$ be the distribution function of dark matter particles. This means that $f(\vec{x},\vec{v},t)d^3xd^3v$ is the number density of particles in an (infinitesimal) phase-space element d^3xd^3v around $\{\vec{x},\vec{v}\}$.

For a collisionless system, we have

$$\frac{df}{dt} = 0. \tag{44}$$

This is the **collisionless Boltzmann equation**, which expresses that in a collisionless system, the phase-space density around each particle is conserved. In other words, there is no diffusion or scattering.

We can now write

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x_i}dx_i + \frac{\partial f}{\partial v_i}dv_i \tag{45}$$

(using Einsteain's summation convenstion, namely that there are sums over dx_i and dv_i), and use

$$\frac{dv_i}{dt} = -\frac{\partial \Phi}{\partial x_i} \tag{46}$$

where Φ is the gravitational potential, to write

$$\frac{\partial f}{\partial t} + v_i\frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0 \tag{47}$$

This equation is also known as **Vlasov equation**, and it has many applications in physics.

The unperturbed distribution is homogeneous in space, namely, independent on $\vec{x}$. While we cannot determine the gravitational potential Φ, we can use Birkhoff's theorm in GR, and take $\Phi = 0$ for this unperturbed solution.

We denote the steady, homogeneous distribution by $f_0(v)$. We further assumed that the unperturbed velocity distribution is isotropic (independent on the direction), and so f_0 depends only on $|\vec{v}|$, as there is no preferred direction.

Let us now add a small perturbation to the dark matter distribution, $f = f_0 + f_1$. The evolution of the perturbation is described by

$$\frac{\partial f_1}{\partial t} + v_i \frac{\partial f_1}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f_0}{\partial v_i} = 0. \tag{48}$$

Note that we neglected the term $\frac{\partial \Phi}{\partial x_i}\frac{\partial f_1}{\partial v_i}$ in Equation 48 as it is of second order.

The gravitational potential obeys Poisson equation,

$$\nabla^2 \Phi = 4\pi G \delta\rho \ , \quad \delta\rho \equiv m \int d^3 v f_1(\vec{x}, \vec{v}, t). \tag{49}$$

where we assumed that m is the mass of a dark matter particle.

Equation 48 is linear. We can therefore look for solutions of the form $f_1 \propto e^{i(\vec{k}\cdot\vec{x} - \omega t)}$. With this choice of f_1 we have

$$\nabla \Phi = \int d^3 x 4\pi G \delta\rho = 4\pi G m \int d^3 x d^3 v f_1(\vec{x}, \vec{v}, t) = -i4\pi G \frac{|\vec{k}|}{k^2}\delta\rho \tag{50}$$

Furthermore,

$$\frac{\partial f_0}{\partial v_i} = \frac{v_i}{|\vec{v}|}\frac{df_0}{d|\vec{v}|}. \tag{51}$$

Using these results and the sultion for f_1 in Equation 48, we get

$$-i\omega f_1 + i\vec{v}\cdot\vec{k} f_1 + i\frac{4\pi G}{k^2}\vec{k}\cdot\vec{v}\frac{1}{|\vec{v}|}\frac{df_0}{d|\vec{v}|}\delta\rho. \tag{52}$$

Multiply by m and integrate over velocities, we find

$$\left[1 - \frac{4\pi G m}{k^2}\int d^3 v \frac{\vec{k}\cdot\vec{v}}{\omega - \vec{k}\cdot\vec{v}}\frac{1}{|\vec{v}|}\frac{df_0}{d|\vec{v}|}\right]\delta\rho = 0. \tag{53}$$

Clearly, non-trivial solutions ($\delta\rho \neq 0$) are obtained for

$$1 - \frac{4\pi G m}{k^2}\int d^3 v \frac{\vec{k}\cdot\vec{v}}{\omega - \vec{k}\cdot\vec{v}}\frac{1}{|\vec{v}|}\frac{df_0}{d|\vec{v}|} = 0 \tag{54}$$

This is nothing but a dispersion equation, which determines a dispersion relation, $\omega = \omega(\vec{k})$ for which this equality is satisfied.

Let us find a solution to this equation in the long wavelength, $\vec{k} \to 0$ limit. We will assume that in this limit, $\omega/|\vec{k}| \to \infty$, so that $\vec{k} \cdot \vec{v}/\omega \ll 1$, and may therefore be consider as a small parameter. This approximation holds as long as the wave numbers are $|\vec{k}| \ll \omega/v_0$, where v_0 is the characteristic velocity of the dask matter particle distribution.

Keeping terms up to second order in $\vec{k} \cdot \vec{v}/\omega$, we may write equation 54 as

$$1 - \frac{4\pi Gm}{k^2\omega^2} \int d^3v \omega^2 \frac{\vec{k} \cdot \vec{v}}{\omega} \left(1 + \frac{\vec{k} \cdot \vec{v}}{\omega}\right) \frac{1}{|\vec{v}|} \frac{df_0}{d|\vec{v}|} = 0 \tag{55}$$

Now comes the trick. The term linear in $\vec{k} \cdot \vec{v}/\omega$ vanishes upon integration, since we integrate over all space, and the rest of the integrand is independent on the direction of $\vec{v}$.

Using integration by parts, the integral of the $(\vec{k} \cdot \vec{v}/\omega)^2$ term, which is $\int d^3v (\vec{k} \cdot \vec{v})^2 |\vec{v}|^{-1} (df_0/d|\vec{v}|)$ gives $-k^2 \int d^3v f_0$ (note that $d/d|\vec{v}|((\vec{k} \cdot \vec{v})^2/|\vec{v}|) = (\vec{k} \cdot \vec{v})^2/|\vec{v}|^2$). This leads to the dispersion relation

$$\omega^2 = -4\pi G\rho, \tag{56}$$

where $\rho = m \int d^3v f_0$ is the number (rather then eenrgy) density.

The time scale for perturbation growth is therefore

$$t \sim \frac{2\pi}{i\omega} = \sqrt{\frac{\pi}{G\rho}}, \tag{57}$$

which is independent on the perturbation scale.

This result has a straight-forward interpretation. Consider a sphere of radius R, mass M and mass density ρ. The acceleration at a point at the edge of the sphere is $g = GM/R^2$, and therefore the time scale for gravitational collapse of the sphere is $\approx \sqrt{2R/g} = \sqrt{2R^3/GM} = \sqrt{\frac{3}{2\pi G\rho}}$. Thus, $1/\sqrt{G\rho}$ is the "free fall time", which is the characteristic time scale for collapse under gravitation in the abscence of processes that resist the collapse (such as pressure).

This result is similar to the results derived above, $\delta \propto a$ in the matter dominated era.

The results we derived, $\omega^2 = -4\pi G\rho$, holds for long wavelengths. Let us next estimate the wavelength where this gravitational instability is suppressed. Such an estimate is obtained by finding the wavelength k_c for which $\omega = 0$. Using Equation 54, we have

$$k_c^2 = -4\pi Gm \int d^3v \frac{1}{|\vec{v}|} \frac{df_0}{d|\vec{v}|} = 4\pi Gm \int d^3v \frac{1}{|\vec{v}|^2} f_0 = 4\pi G\rho\overline{v^{-2}}. \tag{58}$$

Here, the overbar denotes average over the distribution function f_0. The corresponding wavelength is

$$\lambda_c \equiv \frac{2\pi}{k_c} = \sqrt{\frac{\pi}{G\rho}} (\overline{v^{-2}})^{-1/2}. \tag{59}$$

This result is similar to that obtained for normal matter. Note, though, the following. The Jeans scale is given by the product of the perturbation growth time and the speed of sound of the fluid. Here, the scale below which perturbation growth is suppressed is given by the perturbation growth time and the characteristic dispersion in the velocities of the dark matter particles, defined as $(\overline{v^{-2}})^{-1/2}$.

7.2. Free streaming

While the effect of pressure leads to oscillations, the velocity dispersion of dark matter particles lead to suppression of perturbations. On scales $\lambda < \lambda_c$, dark matter particles propagate a distance larger than λ on perturbation growth time scale. During matter domination era, this "free streaming" of particles will "smear out" and erase the perturbations, in a similar way to how photon diffusion suppresses perturbations on scales $\lambda < \lambda_d$ during radiation domination.

The free streaming scale is defined as the distance that dark matter particles propagate on a gravitational perturbation growth time scale, $\lambda_{FS} \equiv a^{-1}(a/\dot{a})v = v/\dot{a}$. Here, v is the characteristic velocity dispersion of dark matter particles. λ_{FS} evolves with time, as $\lambda_{FS} \propto va$ in radiation dominated era, and $\lambda_{FS} \propto va^{1/2}$ in matter dominated era.

We now discriminate between two cases. At early epoch, $a < a_{NR} < a_{\rm eq.}$, dark matter particles are relativistic, $v \approx c$ and $\lambda_{FS} \propto a$. By assumption, dark matter particles become non-raltivisitc at $a_{NR} < a_{\rm dec.}$. At $a > a_{NR}$, when the dark matter particles are already decoupled from the photons (namely, at $a > a_{\rm dec.,DM}$), the redshift of dark matter particles momentum implies $v \propto a^{-1}$ and $\lambda_{FS} \propto a^0$ (rad. dominated) and $\lambda_{FS} \propto a^{-1/2}$ (matter dominated). If the dark matter particles are already non-relativistic, but are still coupled to the photons (i.e., $a_{NR} < a < a_{\rm dec.,DM}$, then $v \propto a^{-1/2}$ and $\lambda_{FS} \propto a^{1/2}$. This can be seen as $v^2 \propto T_{DM} \sim T_r$, and $T_r \propto \rho_r^{1/4} \propto a^{-1}$.

This free streaming will erase perturbations on scale $\lambda < \lambda_{FS}(a = a_{NR}) = c/\dot{a}(a = a_{NR})$.

8. Summary: the resulting perturbed spectrum

We now have all the ingredients to look at the evolution of a perturbation from an initial time $t_i \ll t_{\rm eq.}$ to some final time, $t > t_{\rm de.}$.

- Perturbations on small scales: $\lambda \ll \max(\lambda_D, \lambda_{FS})$ are suppressed by the diffusion of

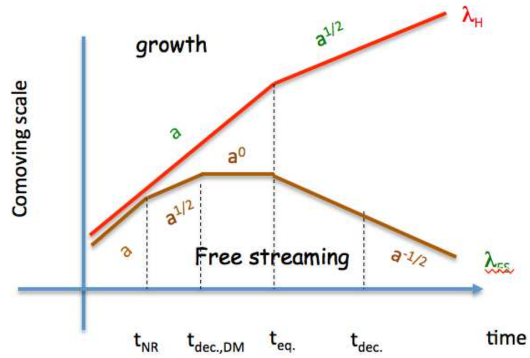

Fig. 5.— In cold dark matter dominated universe, free streaming of dark matter particles erase perturbation growth on scale $\lambda < \lambda_{FS}$.

photons and free streaming of dark matter particles. Thus,

$$\delta(t) \approx 0 \quad \lambda < \max(\lambda_{FS}, \lambda_D). \tag{60}$$

- Perturbations on the large scale, at wavelength $\lambda > \lambda_{\rm eq.}$ grow as a^2 during radiation domination and as a^1 during matter domination. As long as $a < a_{\rm dec.} \ll 1$, we can approximate $\delta \propto a$ for matter dominated phase.

  Thus, these perturbations are amplified by a factor $(a_{\rm eq.}/a_i)^2(a_{\rm dec.}/a_{\rm eq.}))$ from their origin at a_i until $a_{\rm dec.}$. For $a > a_{\rm dec}$, we showd that these perturbations will grow as $\delta \propto D(a) \propto a^\alpha$, with $\alpha < 1$ depends on the cosmology.

- Perturbation on intermediate scale: $\max(\lambda_{FS}, \lambda_D) < \lambda < \lambda_{\rm eq.}$. These perturbations enter the horizon (i.e., their eavelength becomes equal to λ_H at $a = a_{\rm ent.} < a_{\rm eq.}$.

  Once they enter the horizon, the perturbation does ntot grow during the radiation dominated era, as the Jeans scale at this time is comparable to the horizon size. Thus, by the time the scale factor is $a = a_{\rm eq.}$, these perturbations are suppressed, compared to perturbations on larger scale, $\lambda > \lambda_{\rm eq.}$ by a factor $(a_{\rm eq.}/a_{\rm ent.})^2$.

  During the time $a_{\rm eq.} < a < a_{\rm dec.}$, the evolution of these perturbations depends on whether or not Ω_m is dominated by dark matter.

  1. If Ω is dominated by dark matter (as we believe today), then dark matter perturbations grow linearly with a during $a_{\rm eq.} < a < a_{\rm dec.}$. Thus, although the baryon perturbations on scale $\lambda < \lambda_{\rm eq.}$ do not grow but rather oscillate, at decoupling

they are no longer supported by photon pressure, and the baryons fall into the dark matter gravitational potential well after decoupling. Thus, in the presence of significant dark matter component these perturbations do grow linearly in this epoch, and the overall suppression at $a = a_{\rm dec.}$ compared to perturbations on large scale remains $(a_{\rm eq.}/a_{\rm ent.})^2$.

2. In the absence of dark matter, baryon perturbations on scale $\lambda_D < \lambda < \lambda_{\rm eq.}$ do not grow during $a_{\rm eq.} < a < a_{\rm dec.}$, but rather continue to oscillate. The overall suppression of these perturbations at $a_{\rm dec.}$ relative to large scale perturbations is $(a_{\rm eq.}/a_{\rm ent.})^2(a_{\rm dec.}/a_{\rm eq.})$

Since $\lambda_H \propto a$ for $a < a_{\rm eq.}$, we can write $a_{\rm ent.} \propto \lambda$, or $a_{\rm ent.} = a_{\rm eq.}(\lambda/\lambda_{\rm eq.})$. This means that in the presence of dark matter, perturbations on scale $\max(\lambda_{FS}, \lambda_D) < \lambda < \lambda_{\rm eq.}$ are suppressed, compared to perturbations on larger scales, by a factor $(\lambda/\lambda_{\rm eq.})^{-2}$.

In the absence of dark matter, perturbations on scale $\lambda_D < \lambda < \lambda_{\rm eq.}$ are suppressed by a factor $(\lambda/\lambda_{\rm eq.})^{-2}(a_{\rm dec.}/a_{\rm eq.})$.

The signatures of the inhomogeneities in the Universe at the time of decoupling are imprinted on the cosmic microwave background (CMB) radiation. Currently, I still do not know if we have the time to discuss it. Basically, comparison of the CMB anisotropy with the observed inhomogeneities in the present universe, i.e. with the large-scale structure (LSS) of the distribution of galaxies, provides, when combined with the results we derived here on the evolution of the spectrum and amplitude of the inhomogeneities, stringent constraints on the cosmological model.

REFERENCES

[1] H. Mo, F. van den Bosch and S. White, *Galaxy Formation and Evolution* (Cambridge), chapter 4.

[2] T. Padmanabhan, *Structure Formation in the Universe* (Cambridge), chapter 4.

[3] E. Kolb and M. Turner, *The Early Universe*, chapter 9.