The Dirac Equation

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This part of the course is based on Refs. [1], [2] and [3].

1. Introduction

So far we have only discussed scalar fields, such that under a Lorentz transformation $x^\mu \to x^{\mu'} = \lambda^{\mu'}{}_\nu x^\nu$ the field transforms as

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x). \tag{1}$$

We have seen that quantization of such fields gives rise to spin 0 particles. However, in the framework of the standard model there is only one spin-0 particle known in nature: the Higgs particle.

Most particles in nature have an intrinsic angular momentum, or **spin**. To describe particles with spin, we are going to look at fields which themselves *transform non-trivially under the Lorentz group.*

In this chapter we will describe the Dirac equation, whose quantization gives rise to Fermionic spin 1/2 particles. To motivate the Dirac equation, we will start by studying the appropriate representation of the Lorentz group.

2. The Lorentz group, its representations and generators

The first thing to realize is that the set of all possible Lorentz transformations form what is known in mathematics as **group**.

By definition, a group G is defined as a set of objects, or operators (known as *the elements of* G) that may be combined, or multiplied, to form a well-defined product in G which satisfy the following conditions:

1. **Closure**: If a and b are two elements of G, then the product ab is also an element of G.

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2. The multiplication is **associative**: $(ab)c = a(bc)$.
3. There is a unit element I in G, such that $Ia = aI = a$ for every element a in G.
4. For every element a in G there is an **inverse element** a^{-1}, such that $aa^{-1} = a^{-1}a = I$.

Clearly, the set of all Lorentz transformations $\Lambda^{\mu}{}_{\nu}$ fulfill these conditions, and thus they form a group, known as the **(inhomogeneous) Lorentz group**, or **Poincaré group**.[1]

While the group is an abstract mathematical entity, it is often very useful to **represent** each group element by a matrix. (Technically, group representations describe abstract groups in terms of linear transformations of vector spaces). This is particularly useful, as the group operation can be represented by matrix multiplication. In fact, this is what we did so far, without calling it as such !. The important point to note is that *there could be more than a single representation for each group element.* We will see the implications of this below.

Let us now return to fields. We already saw an example of a field that transforms non-trivially under Lorentz transformation - this is the vector field $A_{\mu}(x)$ of electromagnetism,

$$A^{\mu}(x) \rightarrow \Lambda^{\mu}{}_{\nu} A^{\nu}(\Lambda^{-1}x). \tag{2}$$

We will deal with this particular field later.

In general, under Lorentz transformation a field can transform as

$$\phi^{a}(x) \rightarrow D[\Lambda]^{a}{}_{b} \phi^{b}(\Lambda^{-1}x), \tag{3}$$

where the ϕ^a, ϕ^b represent different fields, and $D[\Lambda]^{a}{}_{b}$ are matrices. The matrices $D[\Lambda]$ form a **representation** of the Lorentz group, namely it reflects all the multiplication properties of the Lorentz transformations. As explained above, this means that (i) $D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2]$; (ii) $D[\mathbf{1}] = \mathbf{1}$; and (iii) $D[\Lambda^{-1}] = D[\Lambda]^{-1}$.

2.1. Generators

The next question is: how do we find the different representations?

In order to answer this question, we first generalize our discussion to continuous groups, known as *Lie groups*. In these groups, the parameters of a product element are analytical

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[1] Lorentz transformations can be split into 3 categories: translations, rotations and boosts. If we omit translations and we consider only rotations and boosts, the resulting group is known as **homogeneous Lorentz group**.

functions (= having derivatives of all orders) of the parameters of the factors. The analytic nature of the functions (differentiability) allows us to develop the concept of **generators**, and reduce the study of the whole group to a study of the group elements in the neighborhood of the identity element.

Roughly speaking, a set of **generators** is a set of group elements (all infinitesimally close to the identity) such that possibly repeated application of the generators on themselves and each other is capable of producing all the elements in the group. **Thus, in order to find all the representations, we look at the *Lie algebra* of infinitesimal transformations.**

Clearly, the Lorentz group is a Lie group, so let's look at infinitesimal transformations of the Lorentz group as a demonstrator. Consider a Lorentz transformation that is infinitesimally close to the identity

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \epsilon\omega^{\mu}{}_{\nu} + \mathcal{O}(\epsilon^2) \tag{4}$$

for infinitesimal ϵ. We saw (see "Classical Fields", Equation 49) that the condition for a Lorentz transformation: $\Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\rho}\eta^{\sigma\rho} = \eta^{\mu\nu}$ is fulfilled (to first order in ϵ) provided that ω is anti-symmetric:

$$\omega^{\mu\nu} + \omega^{\nu\mu} = 0. \tag{5}$$

Note that an antisymmetric 4×4 matrix has $4 \times 3/2 = 6$ independent components, which agrees with the 6 transformations of the Lorentz group: 3 rotations and 3 boosts.

Let us now introduce a basis of these infinitesimal Lorentz transformations, namely a **basis of these six 4×4 anti-symmetric matrices**. We could call them $(\mathcal{M}^{A})^{\mu\nu}$, with $A = 1, ..., 6$. But in fact, it's going to turn out better (although initially a little confusing) to replace the single index A with a pair of antisymmetric indices $[\rho\sigma]$, where $\rho, \sigma = 0, ..., 3$. We thus call our six basis matrices $(\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu}$.

The antisymmetry on the ρ and σ indices means that, for example, $\mathcal{M}^{01} = -\mathcal{M}^{10}$, etc. Recall again that ρ **and** σ **label six different matrices**. The indices μ and ν label the components of each matrix. Note that the matrices are also antisymmetric on the $\mu\nu$ indices because each one is by itself an antisymmetric matrices.

With this notation in place, we can write a basis of six 4×4 antisymmetric matrices as (at the moment, we just pulled it out of a hat)

$$(\mathcal{M}^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}, \tag{6}$$

where the indices μ and ν are those of the 4×4 matrix, while ρ and σ denote which basis element we are dealing with.

If we use these matrices for anything practical (for example, if we want to multiply them together, or act on some field) we will typically need to lower one index, so we have

$$(\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu} = \eta^{\rho\mu}\delta^{\sigma}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}_{\nu}. \tag{7}$$

Since we lowered the index with the Minkowski metric, we pick up various minus signs which means that when written in this form, the matrices are no longer necessarily antisymmetric.

Plugging directly in Equation 7, we can give two examples of these basis matrices:

$$(\mathcal{M}^{01})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (\mathcal{M}^{12})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

The first, $\mathcal{M}^{01}$, generates (infinitesimal) boosts in the x^1 direction. It is real and symmetric. The second, $\mathcal{M}^{12}$, generates (infinitesimal) rotations in the (x^1, x^2)-plane. It is real and anti-symmetric.

Being the basis of anti-symmetric matrices, we can now write any $\omega^{\mu}{}_{\nu}$ as a linear combination of $\mathcal{M}^{\rho\sigma}$,

$$\omega^{\mu}{}_{\nu} = \frac{1}{2}\Omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu}, \tag{9}$$

where $\Omega_{\rho\sigma}$ are just six numbers (again, we use the same notation, namely antisymmetric in the indices), that tell us what Lorentz transformation we are doing.

The six basis matrices $\mathcal{M}^{\rho\sigma}$ are the **generators** of the Lorentz transformations. **They obey the Lorentz Lie algebra of Lorentz transformations**,

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = \eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} + \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau} \tag{10}$$

where we have suppressed the matrix indices. *It should be stressed that these commutation relations encapsulates all that is important about the Lorentz group; Any matrices that obey these commutations relations form a representation of the Lorentz group.* (You may be familiar with that from SU(2) rotation group in QM). So what we really do is find matrices which obey this commutation relations - the Lie algebra - and then we can use them to express any Lorentz transformation.

The transformations $\omega^{\mu}{}_{\nu}$ are infinitesimally close to unity. We can express any **finite** Lorentz transformation as the exponential[2]

$$\Lambda = \exp(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}). \tag{11}$$

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[2]This follows from the following: we first write $\omega = \epsilon\tilde{\omega}$. Let $\epsilon = 1/n$. Any finite transformation can be written as $\Lambda = \lim_{n\to\infty}(\delta + (1/n)\tilde{\omega})^n$.

Let me stress again what each of these objects are: the $\mathcal{M}^{\rho\sigma}$ are six 4×4 basis elements of the Lorentz group; the $\Omega_{\rho\sigma}$ are six numbers telling us what kind of Lorentz transformation we are doing (for example, they say things like rotate by $\theta = \pi/7$ about the x^3-direction and run at speed $v = 0.2$ in the x^1 direction).

3. The Spinor Representation

Having discussed (briefly) the generators of the Lorentz group, let us return to our original question: how do we find the different representations of the Lorentz group ?

Basically, we need to find matrices which satisfy the Lorentz algebra commutation relations, Equation 10.

We are going to construct the **spinor representation**. To do this, we start by defining something which, at first sight, has nothing to do with the Lorentz group.

3.1. Clifford Algebra

Any set of objects γ^μ which obey the **anti-commutation** relations is said to form a **Clifford algebra**, namely

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\nu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}, \tag{12}$$

where γ^μ with $\mu = 0, 1, 2, 3$ are a set of four matrices, and the $\mathbf{1}$ on the right hand side denotes the unit matrix. This means that we are looking for 4 matrices such that

$$\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \text{for} \quad \mu \neq \nu, \tag{13}$$

and

$$(\gamma^0)^2 = \mathbf{1} \quad , \quad (\gamma^i)^2 = -\mathbf{1} \quad i = 1, 2, 3. \tag{14}$$

It's not hard to convince yourself that there are <u>no representations</u> of the Clifford algebra using 2×2 or 3×3 matrices. The simplest representation of the Clifford algebra is in terms of 4×4 matrices. There are many such examples of 4×4 matrices which obey Equation 12. For example, we may take

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{15}$$

where each element is itself a 2×2 matrix, with the σ^i the **Pauli matrices**,

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{16}$$

It is easy to check that these matrices themselves satisfy $\{\sigma^i, \sigma^j\} = 2\delta^{ij}$.

One can construct many other representations of the Clifford algebra by taking $V\gamma^\mu V^{-1}$ for any invertible matrix V. However, up to this equivalence, one can prove that there is a unique irreducible representation of the Clifford algebra. The matrices in Equation 15 provide one example, known as the **Weyl** or **chiral representation** (for reasons that will soon become clear). We will soon restrict ourselves further, and consider only representations of the Clifford algebra that are related to the chiral representation by a unitary transformation V.

So what does the Clifford algebra have to do with the Lorentz group? As we are going to show shortly, the Lorentz group can be represented by the six matrices $S^{\rho\sigma}$, which are defined by the commutator of two γ^μ,

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \left\{ \begin{array}{cc} 0 & \rho = \sigma \\ \frac{1}{2}\gamma^\rho\gamma^\sigma & \rho \neq \sigma \end{array} \right\} = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma} . \tag{17}$$

(You can verify this result directly, using the anti-commutation relations of the σ matrices). Clearly, $S^{\mu\nu} = -S^{\nu\mu}$, resulting in six such matrices.

3.1.1. Algebraic properties of $S^{\mu\nu}$ matrices.

Claim [1]. $[S^{\mu\nu}, \gamma^\rho] = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu}$.

Proof. For $\mu \neq \nu$ we have

$$\begin{aligned} [S^{\mu\nu}, \gamma^\rho] &= \tfrac{1}{2}[\gamma^\mu\gamma^\nu, \gamma^\rho] \\ &= \tfrac{1}{2}\gamma^\mu\gamma^\nu\gamma^\rho - \tfrac{1}{2}\gamma^\rho\gamma^\mu\gamma^\nu \\ &= \tfrac{1}{2}\gamma^\mu \{\gamma^\nu, \gamma^\rho\} - \tfrac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu - \tfrac{1}{2}\{\gamma^\rho, \gamma^\mu\}\gamma^\nu + \tfrac{1}{2}\gamma^\mu\gamma^\rho\gamma^\nu \\ &= \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\rho\mu} . \end{aligned}$$

Claim [2]. **The matrices $S^{\mu\nu}$ form a representation of the Lorentz algebra** (Equation 10), in the sense that

$$[S^{\mu\nu}, S^{\rho\sigma}] = \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} + \eta^{\mu\sigma}S^{\nu\rho} - \eta^{\nu\sigma}S^{\mu\rho} \tag{18}$$

Proof. Taking $\rho \neq \sigma$ and using the previous claim proven above, we have

$$\begin{aligned} [S^{\mu\nu}, S^{\rho\sigma}] &= \tfrac{1}{2}[S^{\mu\nu}, \gamma^\rho\gamma^\sigma] \\ &= \tfrac{1}{2}[S^{\mu\nu}, \gamma^\rho]\gamma^\sigma + \tfrac{1}{2}\gamma^\rho[S^{\mu\nu}, \gamma^\sigma] \\ &= \tfrac{1}{2}\gamma^\mu\gamma^\sigma\eta^{\nu\rho} - \tfrac{1}{2}\gamma^\nu\gamma^\sigma\eta^{\rho\mu} + \tfrac{1}{2}\gamma^\rho\gamma^\mu\eta^{\nu\sigma} - \tfrac{1}{2}\gamma^\rho\gamma^\nu\eta^{\sigma\mu} \end{aligned} \tag{19}$$

We can use now Equation 17 to write $\gamma^\mu\gamma^\sigma = 2S^{\mu\sigma} + \eta^{\mu\sigma}$, and get

$$[S^{\mu\nu}, S^{\rho\sigma}] = S^{\mu\sigma}\eta^{\nu\rho} - S^{\nu\sigma}\eta^{\rho\mu} + S^{\rho\mu}\eta^{\nu\sigma} - S^{\rho\nu}\eta^{\sigma\mu} \tag{20}$$

which completes the proof.

3.2. Dirac Spinors

The γ^μ are 4×4 matrices, and therefore the $S^{\mu\nu}$ are also 4×4 matrices. We will call the indices of the rows and columns of these matrices $\alpha, \beta = 1, 2, 3, 4$.[3]

The matrices $(S^{\mu\nu})^\alpha{}_\beta$ act on a field. We introduce the **Dirac spinor field** $\psi^\alpha(x)$, which is an object with four complex components, which we label by $\alpha = 1, 2, 3, 4$. Under Lorentz transformation, we have

$$\psi^\alpha(x) \rightarrow S[\Lambda]^\alpha{}_\beta \psi^\beta(\Lambda^{-1}x), \tag{21}$$

where

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) \tag{22}$$

and

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right). \tag{23}$$

(Again: $\mathcal{M}^{\rho\sigma}$ are the generators of the Lorentz transformation Λ which transforms the coordinates x, while $S^{\rho\sigma}$ are the generators of the representation of Lorentz algebra which determines the transformation of the fields ψ).

Although the basis of the generators $\mathcal{M}^{\rho\sigma}$ and $S^{\rho\sigma}$ are different, we use the same six numbers $\Omega_{\rho\sigma}$ in both Λ and $S[\Lambda]$. This ensures that we are doing the same Lorentz transformation on x and ψ. Note that we denote both the generator $S^{\rho\sigma}$ and the full Lorentz transformation $S[\Lambda]$ as "S". To avoid confusion, the latter will always come with the square brackets $[\Lambda]$.

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[3]We don't use the labeling 0,1,2,3, as these have got nothing to do with space and time.

Both Λ and $S[\Lambda]$ are 4×4 matrices. So how can we be sure that the spinor representation is something new, and isn't equivalent to the familiar representation $\Lambda^\mu{}_\nu$? To see that the two representations are truly different, let's look at some specific transformations.

Example I: Rotations.
The generators S^{ij} become

$$S^{ij} = \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = -\frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \quad \text{(for i} \neq \text{j)} \tag{24}$$

If we define the rotation parameters as $\Omega_{ij} = -\epsilon_{ijk}\varphi^k$ (namely, $\Omega_{12} = -\varphi^3$, etc.), then the rotation matrix becomes

$$S[\Lambda] = \exp(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}) = \begin{pmatrix} e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{25}$$

where we need to remember that $\Omega_{12} = -\Omega_{21} = -\varphi^3$ when following factors of 2.

Consider as an example a rotation by 2π about the x^3-axis. This is achieved by $\vec{\varphi} = (0, 0, 2\pi)$. The spinor rotation matrix becomes

$$S[\Lambda] = \begin{pmatrix} e^{+i\pi\sigma^3} & 0 \\ 0 & e^{+i\pi\sigma^3} \end{pmatrix} = -\mathbf{1}. \tag{26}$$

Thus, under a 2π rotation we get

$$\psi^\alpha(x) \to -\psi^\alpha(x) \tag{27}$$

which is **not the same** as the transformation law of a vector!. There is this extra minus sign (which connects to the Fermi-Dirac statistics, as we will see later).

To check that we haven't been cheating with factors of 2, let's see how a vector would transform under a rotation by $\vec{\varphi} = (0, 0, \varphi^3)$. We have:

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \varphi^3 & 0 \\ 0 & -\varphi^3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{28}$$

(with $\Omega_{12} = -\varphi^3$). Thus, when we rotate a vector by $\varphi^3 = 2\pi$, we learn that $\Lambda = 1$, as one expects. We therefore conclude that $S[\Lambda]$ is definitely a different representation from the familiar vector representation $\Lambda^\mu{}_\nu$.

Example II: Boosts.

$$S^{0i} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{29}$$

Writing the boost parameter as $\Omega_{i0} = -\Omega_{0i} = \chi_i$, we have

$$S[\Lambda] = \begin{pmatrix} e^{+\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{30}$$

Representations of the Lorentz group are not unitary.
For the rotations given in Equation 25, $S[\Lambda]$ is unitary, namely $S[\Lambda]^\dagger S[\Lambda] = \mathbf{1}$. However, for the boosts given in Equation 30, $S[\Lambda]$ is not unitary (there was no "i" in the exponent).

In fact, there are **no** finite dimensional unitary representations of the Lorentz group. We have demonstrated this explicitly for the spinor representation using the chiral representation (Equation 15) of the Clifford algebra.

We can get a feel for why it is true for a spinor representation constructed from any representation of the Clifford algebra. Recall that

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$$

and therefore the representation is unitary if $S^{\mu\nu}$ are anti-hermitian, i.e., $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$. However, this can never happen: using equation 17 we have

$$(S^{\mu\nu})^\dagger = -\frac{1}{4}[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger], \tag{31}$$

which can be anti-hermitian if all γ^μ are hermitian or all are anti-hermitian. This, though, can never happen since

$$\begin{aligned} (\gamma^0)^2 = \mathbf{1} \quad &\Rightarrow \quad \text{Real Eigenvalues} \\ (\gamma^i)^2 = -\mathbf{1} \quad &\Rightarrow \quad \text{Imaginary Eigenvalues} \end{aligned} \tag{32}$$

So we could pick γ^0 to be hermitian, but we can only pick γ^i to be anti-hermitian. Indeed, in the chiral representation (Equation 15), the matrices have this property: $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. This can be traced back to the different signs in time and space in Minkowski space-time. In general there is no way to pick γ^μ such that $S^{\mu\nu}$ are anti-hermitian.

4. Constructing an Action for the Dirac Spinor Field

We now completed our task and found a representation of the Lorentz group. The representation we found work on a new field: the Dirac spinor field, ψ, introduced in Equation 21. Our next goal is to construct a Lorentz invariant equation of motion. We do this by constructing a **Lorentz invariant action**.

We begin in a naive way which will not work, but will give us a clue how to proceed. Define

$$\psi^\dagger(x) = (\psi^\star)^T(x) \tag{33}$$

which is the usual adjoint of a multi-component object. We could then try to form a Lorentz scalar by taking the product $\psi^\dagger\psi$, with the spinor indices summed over. Let's see how this transforms under Lorentz transformations,

$$\begin{array}{rcl} \psi(x) & \to & S[\Lambda]\psi(\Lambda^{-1}x) \\ \psi^\dagger(x) & \to & \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger. \end{array} \tag{34}$$

Thus, $\psi^\dagger(x)\psi(x) \to \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger S[\Lambda]\psi(\Lambda^{-1}x)$. However, we have seen that for some Lorentz transformations $S[\Lambda]^\dagger S[\Lambda] \neq 1$, since the representation is not unitary. This means that $\psi^\dagger\psi$ is not good: it doesn't have any nice transformation under the Lorentz group, and certainly **it is not a Lorentz scalar.** Thus, the fact that $S[\Lambda]$ is not unitary is what failed our attempt.

But now we see why it fails, we can also see how to solve the problem. Let's pick a representation of the Clifford algebra which, like the chiral representation (Equation 15), satisfies $(\gamma^0)^\dagger = \gamma^0$ and $(\gamma^i)^\dagger = -\gamma^i$. Then, for all $\mu = 0, 1, 2, 3$ we have:

$$\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger \tag{35}$$

which, in turn, gives

$$(S^{\mu\nu})^\dagger = \frac{1}{4}\left[(\gamma^\nu)^\dagger, (\gamma^\mu)^\dagger\right] = -\gamma^0 S^{\mu\nu}\gamma^0, \tag{36}$$

and thus we have

$$S[\Lambda]^\dagger = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\dagger\right) = \gamma^0 S[\Lambda]^{-1}\gamma^0. \tag{37}$$

Thus, the reason why we failed producing a Lorentz scalar out of $\psi^\dagger(x)\psi(x)$ is because $S[\Lambda]^\dagger \neq S[\Lambda]^{-1}$, but $S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma^0$.

With this in mind, we can now *define* the **Dirac adjoint** (or **Dirac Conjugate**),

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0. \tag{38}$$

Let us now see what Lorentz covariant objects we can form out of a Dirac spinor ψ and its adjoint $\bar{\psi}$.

Claim [3]: $\bar{\psi}\psi$ is a Lorentz scalar.

Proof. under a Lorentz transformation,

$$\begin{aligned} \bar{\psi}(x)\psi(x) &= \psi^\dagger(x)\gamma^0\psi(x) \\ &\rightarrow \psi^\dagger(\Lambda^{-1}x)S[\Lambda]^\dagger\gamma^0 S[\Lambda]\psi(\Lambda^{-1}x) \\ &= \psi^\dagger(\Lambda^{-1}x)\gamma^0\psi(\Lambda^{-1}x) \\ &= \bar{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x), \end{aligned} \tag{39}$$

which is indeed the transformation law of a Lorentz scalar.

Claim [4]: $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector, namely

$$\bar{\psi}(x)\gamma^\mu\psi(x) \rightarrow \Lambda^\mu{}_\nu\bar{\psi}(\Lambda^{-1}x)\gamma^\nu\psi(\Lambda^{-1}x). \tag{40}$$

This equation implies that we can treat the $\mu = 0, 1, 2, 3$ index on the γ^μ matrices as a true vector index. In particular we can form Lorentz scalars by contracting it with other Lorentz indices.

Proof. Under Lorentz transformation we have

$$\bar{\psi}\gamma^\mu\psi \rightarrow \bar{\psi}S[\Lambda]^{-1}\gamma^\mu S[\Lambda]\psi. \tag{41}$$

Thus, if $\bar{\psi}\gamma^\mu\psi$ is to transform as a vector, we must have

$$S[\Lambda]^{-1}\gamma^\mu S[\Lambda] = \Lambda^\mu{}_\nu\gamma^\nu. \tag{42}$$

Let us show this. Infinitesimally, we have

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) \approx 1 + \frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma} + ... \tag{43}$$

and

$$S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right) \approx 1 + \frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma} + ... \tag{44}$$

and thus the requirement in Equation 42 becomes

$$-[S^{\rho\sigma}, \gamma^\mu] = (\mathcal{M}^{\rho\sigma})^\mu{}_\nu\gamma^\nu \tag{45}$$

(omitting the α, β indices on γ^μ and $S^{\mu\nu}$, but otherwise leaving all other indices explicit).

Equation 45 follows claim [1], $[S^{\rho\sigma}, \gamma^\mu] = \gamma^\rho \eta^{\sigma\mu} - \gamma^\sigma \eta^{\mu\rho}$. This can be seen by first expanding $\mathcal{M}$ in the right hand side of Equation 45 using Equation 7:

$$\begin{aligned} (\mathcal{M}^{\rho\sigma})^\mu{}_\nu \gamma^\nu &= (\eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu)\gamma^\nu \\ &= \eta^{\rho\mu}\gamma^\sigma - \eta^{\sigma\mu}\gamma^\rho. \end{aligned} \tag{46}$$

But in claim [1] we showed that the right hand side of Equation 46 is just equal to $-[S^{\rho\sigma}, \gamma^\mu]$, which proves Equation 45, and completes the proof that $\bar{\psi}\gamma^\mu\psi$ transforms as a vector.

Claim [5]. $\bar{\psi}\gamma^\mu\gamma^\nu\psi$ transforms as a Lorentz tensor. More precisely, the symmetric part is a Lorentz scalar, proportional to $\eta^{\mu\nu}\bar{\psi}\psi$, while the antisymmetric part is a Lorentz tensor, proportional to $\bar{\psi}S^{\mu\nu}\psi$.

The proof of this claim follows the same steps as above.

Following the above discussion, we have now the three bilinears of the Dirac field, $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\gamma^\mu\gamma^\nu\psi$, each of which transforms covariantly under the Lorentz group. We can now build **a Lorentz invariant action** out of these; in fact, we need only the first two. We choose

$$S = \int d^4x \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x). \tag{47}$$

This is **Dirac action**. ($\bar{\psi}\psi$ is a Lorentz scalar; $\bar{\psi}\gamma^\mu\psi$ is a Lorentz vector, so contracting it with ∂_μ produces another scalar). The factor i is added to make the action real; upon complex conjugation, it cancels a minus sign that comes from integration by parts.

As we will show shortly, after quantization this theory describes particles and anti-particles of mass $|m|$ and spin 1/2. Notice that the Lagrangian is first order, rather than the second order Lagrangians we were working with for scalar fields. Also, the mass appears in the Lagrangian as m, which can be positive or negative.

5. The Dirac Equation

The Equation of motion follows the action in Equation 47 by varying with respect to ψ and $\bar{\psi}$ independently. Varying with respect to $\bar{\psi}$ gives

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0.} \tag{48}$$

This is the **Dirac Equation**.

Varying with respect to ψ gives the conjugate equation

$$i\partial_\mu\bar{\psi}\gamma^\mu - m\bar{\psi} = 0 \tag{49}$$

The Dirac equation is *first order in derivatives*, yet (miraculously) Lorentz invariant.

Note that if we tried to write down a first order equation of motion for a scalar field, it would have to look like $v^\mu \partial_\mu \phi = ...$, which necessarily includes a privileged vector in spacetime v^μ and is not Lorentz invariant. However, for spinor fields, the magic of the γ^μ matrices implies that the Dirac Lagrangian is Lorentz invariant.

The Dirac Equation mixes up different components of ψ through the matrices γ^μ. Interestingly, each individual component itself solves the Klein-Gordon Equation. To see this, we write

$$(i\gamma^\nu \partial_\nu + m)(i\gamma^\mu \partial_\mu - m)\psi = -(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi = 0, \tag{50}$$

and use the fact that $\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu \partial_\nu = \partial_\mu \partial^\mu$, to get

$$-(\partial_\mu \partial^\mu + m^2)\psi = 0. \tag{51}$$

There are no γ^μ matrices in the last equation, and so it applies to each component ψ^α with $\alpha = 1, 2, 3, 4$.

The Slash Notation

We often encounter 4-vectors contracted with γ^μ matrices. Following Feynman, we define:

$$\not{A} \equiv \gamma^\mu A_\mu \tag{52}$$

Using this notation, the Dirac Equation can be written as

$$(i\not{\partial} - m)\psi = 0. \tag{53}$$

6. Chiral Spinors

So far, when we needed an explicit form of the γ^μ matrices, we have used the chiral representation given in Equation 15:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

In this representation, the spinor rotation transformation $S[\Lambda_{\text{rot}}]$ and boost transformation $S[\Lambda_{\text{boost}}]$ were computed in Equations 25 and 30, respectively. Both are *block diagonal*:

$$S[\Lambda_{\text{rot}}] = \begin{pmatrix} e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{+i\vec{\varphi}\cdot\vec{\sigma}/2} \end{pmatrix} \quad \text{and} \quad S[\Lambda_{\text{boost}}] = \begin{pmatrix} e^{+\vec{\chi}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{-\vec{\chi}\cdot\vec{\sigma}/2} \end{pmatrix} \tag{54}$$

This means that the Dirac spinor representation of the Lorentz group is **reducible**: it decomposes into two **irreducible** representations, acting separately on two-component spinors $u_\pm$.[4] In the chiral representation, $u_\pm$ are defined by

$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \tag{55}$$

The two-component objects $u_\pm$ are called **Weyl spinors**, or **chiral spinors**. (In some textbooks, you find a different notation, u_+ (u_-) written as ψ_L (ψ_R)). They transform in the same way under rotations:

$$u_\pm \to e^{i\vec{\varphi}\cdot\vec{\sigma}/2} u_\pm, \tag{56}$$

but oppositely under boosts:

$$u_\pm \to e^{\pm\vec{\chi}\cdot\vec{\sigma}/2} u_\pm \tag{57}$$

In group theory language, u_+ is in the $(\frac{1}{2}, 0)$ representation of the Lorentz group, while u_- is in the $(0, \frac{1}{2})$ representation. The Dirac spinor ψ lies in the $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ representation.

6.1. The Weyl Equation

Let us see how the decomposition in Equation 55 into Weyl spinors affects the Dirac Lagrangian. Using Equations 55, 38 and 15 we can write

$$\begin{aligned} \mathcal{L} &= \bar{\psi}(i\partial\!\!\!/ - m)\psi = (u_+^\dagger, u_-^\dagger) \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \left[i \begin{pmatrix} 0 & \partial_0 + \sigma^i \partial_i \\ \partial_0 - \sigma^i \partial_i & 0 \end{pmatrix} - m \right] \begin{pmatrix} u_+ \\ u_- \end{pmatrix} \\ &= i u_-^\dagger \sigma^\mu \partial_\mu u_- + i u_+^\dagger \bar{\sigma}^\mu \partial_\mu u_+ - m(u_+^\dagger u_- + u_-^\dagger u_+) = 0 \end{aligned} \tag{58}$$

where we have used the notation

$$\sigma^\mu = (\mathbf{1}, \sigma^i) \quad \text{and} \quad \bar{\sigma}^\mu = (\mathbf{1}, -\sigma^i) \tag{59}$$

to describe the Pauli matrices with $\mu = 0, 1, 2, 3$.

From Equation 58, we find that for a *massive* fermion, both u_+ and u_- are required, since they couple through the mass term. However, for a *massless* fermion the coupling vanishes, and only u_+ (or u_-) is required for the equation of motion, which becomes

$$\begin{aligned} & i\bar{\sigma}^\mu \partial_\mu u_+ = 0; \\ \text{or} \quad & i\sigma^\mu \partial_\mu u_- = 0. \end{aligned} \tag{60}$$

---

[4]Note that the γ^μ matrices themselves were *irreducible* representations of the Clifford algebra; however, we constructed from them a *reducible* representation of the Lorentz group.

These are known as **Weyl equations**. Note that in the standard model of particle physics, **all** fermions are massless, and therefore they are described by the Weyl spinors; Particles obtain their mass by the *Higgs mechanism*, which, unfortunately, is learned only in a more advanced course on the standard model. We also allow ourself to call these particles "Fermions", although we haven't shown it yet; though, when we will quantize the Dirac spinor in the next chapter, we will see that it gives rise to spin-1/2 particles.

Degrees of Freedom

Let us look at the degrees of freedom in a spinor. First note that the Dirac fermion (=Dirac spinor field) has 4 complex components = 8 real components (see Equation 21).

How do we count degrees of freedom? In classical mechanics, the number of degrees of freedom of a system is equal to the dimension of the configuration space or, equivalently, half the dimension of the phase space. In field theory we have an infinite number of degrees of freedom, but it makes sense to count the number of degrees of freedom per spatial point: this should at least be finite. For example, in this sense a real scalar field ϕ has a single degree of freedom. At the quantum level, this translates to the fact that it gives rise to a single type of particle. A classical complex scalar field has two degrees of freedom, corresponding to the particle and the anti-particle in the quantum theory.

What about the Dirac spinor ? One might think that there are 8 degrees of freedom. But this isn't right. Crucially, and in contrast to the scalar field, the equation of motion is first order rather than second order. In particular, for the Dirac Lagrangian, the momentum conjugate to the spinor ψ is given by

$$\pi_\psi = \partial\mathcal{L}/\partial\dot{\psi} = i\psi^\dagger. \tag{61}$$

It is not proportional to the time derivative of ψ. This means that the phase space for a spinor is therefore parameterized by ψ and $\psi^\dagger$, while for a scalar it is parameterized by ϕ and $\pi = \dot{\phi}$. So the phase space of the Dirac spinor ψ has 8 real dimensions and correspondingly the number of real degrees of freedom is 4. We will see in the next chapter that, in the quantum theory, this counting manifests itself as two degrees of freedom (spin up and down) for the particle, and a further two for the anti-particle.

A similar counting for the Weyl fermion tells us that it has two degrees of freedom.

6.2. The γ^5 matrix

We have seen that the Lorentz group matrices $S[\Lambda]$ came out to be block diagonal in Equation 54. This is due to our choice of the specific representation in Equation 15. In fact, this is why the representation in Equation 15 is called the chiral representation: it's because the decomposition of the Dirac spinor ψ is simply given by Equation 55.

But we could choose a different representation γ^μ of the Clifford algebra, so that

$$\gamma^\mu \to U\gamma^\mu U^{-1} \quad \text{and} \quad \psi \to U\psi. \tag{62}$$

In this representation, $S[\Lambda]$ will not be block diagonal. However, we can still define the chiral spinors in an invariant way, by introducing the "fifth" gamma matrix,

$$\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{63}$$

(Note that by adding the (-) sign, I follow David Tong's convention, rather than Peskin and Schroeder). You can check that this matrix satisfies

$$\{\gamma^5, \gamma^\mu\} = 0 \quad \text{and} \quad (\gamma^5)^2 = +1. \tag{64}$$

(The reason why it is called γ^5 is because the set of matrices $\tilde{\gamma}^A = (\gamma^\mu, i\gamma^5)$ with $A = 0, 1, 2, 3, 4$ satisfy the five-dimensional Clifford algebra, $\{\tilde{\gamma}^A, \tilde{\gamma}^B\} = 2\eta^{AB}$).

You can also check that $[S_{\mu\nu}, \gamma^5] = 0$, which means that γ^5 is a scalar under rotations and boosts. Since $(\gamma^5)^2 = 1$, this means that we may form the Lorentz invariant **projection operators**

$$P_\pm = \frac{1}{2}(1 \pm \gamma^5), \tag{65}$$

such that $P_+^2 = P_+$, $P_-^2 = P_-$ and $P_+P_- = 0$. Further, one can check by direct substitution in Equation 63 that for the chiral representation (Equation 15),

$$\gamma^5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}, \tag{66}$$

from which we see that the operators $P_\pm$ project onto the Weyl spinors, $u_\pm$. However, for an arbitrary representation of the Clifford algebra, we may use γ^5 to define the **chiral spinors**,

$$\psi_\pm = P_\pm \psi, \tag{67}$$

which form the irreducible representations of the Lorentz group. ψ_+ is often called a "right-handed" spinor, while ψ_- is "left-handed". These are thus the analogues of the Weyl spinors, but in arbitrary basis. In particular, in the chiral representation, one finds

$$\psi_+ = \begin{pmatrix} u_+ \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_- = \begin{pmatrix} 0 \\ u_- \end{pmatrix}. \tag{68}$$

6.3. Discrete Symmetries of the Dirac Theory: Parity

The spinors $\psi_\pm$ are related to each other by *parity*. Let us first define this concept.

The Lorentz group (=the group of all Lorentz transformations) is defined by $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, such that

$$\Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma \eta^{\nu\sigma} = \eta^{\mu\rho}. \tag{69}$$

So far we have only considered **continuous** transformations Λ, which are continuously connected to the identity. Only these transformations have an infinitesimal form. However there are also two **discrete symmetries** which are part of the Lorentz group:

$$\begin{array}{ll} \text{Time reversal} & T : x^0 \to -x^0 \ ; \quad x^i \to x^i \\ \text{Parity} & P : x^0 \to x^0 \ ; \quad x^i \to -x^i \end{array} \tag{70}$$

We are not going to discuss much time reversal in this course (see Peskin & Schroeder for further discussion). However, we do want to discuss **parity**, as it plays an important role in the standard model (and in particular in the theory of weak interactions). As we saw, parity sends $(t, \vec{x})$ to $(t, -\vec{x})$, reversing the *handedness* of space.

Under parity, the left and right-handed spinors are exchanged. This follows from the transformation of the spinors under the Lorentz group. In the chiral representation, we saw that the rotation (Equation 56) and boost (Equation 57) transformations for the Weyl spinors $u_\pm$ are

$$u_\pm \overset{rot}{\to} e^{i\vec{\varphi}\cdot\vec{\sigma}/2} u_\pm \quad \text{and} \quad u_\pm \overset{boost}{\to} e^{\pm\vec{\chi}\cdot\vec{\sigma}/2} u_\pm \tag{71}$$

Under parity, both the spatial coordinates and the momentum change sign, and hence rotations don't change sign. But boosts do flip sign. This confirms that parity exchanges right-handed and left-handed spinors, $P : u_\pm \to u_\mp$, or in the notation $\psi_\pm = \frac{1}{2}(1 \pm \gamma^5)\psi$, we have

$$P : \psi_\pm(t, \vec{x}) \to \psi_\mp(t, -\vec{x}). \tag{72}$$

Using this knowledge of how chiral spinors transform, and the fact that $P^2 = 1$, we see that the action of parity on the Dirac spinor itself can be written as

$$P : \psi(\vec{x}, t) \to \gamma^0 \psi(-\vec{x}, t). \tag{73}$$

Also, note that if $\psi(\vec{x}, t)$ satisfies the Dirac equation, then the parity transformed spinor $\gamma^0\psi(-\vec{x}, t)$ also satisfies the Dirac equation:

$$(i\gamma^0\partial_t + i\gamma^i\partial_i - m)\gamma^0\psi(-\vec{x}, t) = \gamma^0(i\gamma^0\partial_t - i\gamma^i\partial_i - m)\psi(-\vec{x}, t) = 0 \tag{74}$$

where the extra minus sign from passing γ^0 through γ^i is compensated by the derivative acting on $-\vec{x}$ instead of $+\vec{x}$.

6.4. Chiral Interactions and Dirac Field Bilinears

We would like to ask now the following question. Consider the general expression $\bar{\psi}\Gamma\psi$, where Γ is any 4×4 constant matrix. Can we decompose this expression into terms that have definite transformation properties under the Lorentz group? Once we know how to do that, we could build Lorentz invariant Lagrangians from these terms.

In order to carry this calculation, let us look at how the different possible interaction terms change under parity. We can look at each of the spinor bilinears from which we built the action,

$$P : \bar{\psi}\psi(\vec{x}, t) \to \bar{\psi}\psi(-\vec{x}, t). \tag{75}$$

This is the transformation under parity we expect from a scalar. Furthermore, we saw in Equation 39 that $\bar{\psi}\psi$ transforms as a scalar under continuous Lorentz transformation.

For the vector $\bar{\psi}\gamma^\mu\psi$, we can look at the temporal and spatial components separately,

$$\begin{aligned} &P : \bar{\psi}\gamma^0\psi(\vec{x}, t) \to \bar{\psi}\gamma^0\psi(-\vec{x}, t); \\ &P : \bar{\psi}\gamma^i\psi(\vec{x}, t) \to \bar{\psi}\gamma^0\gamma^i\gamma^0\psi(-\vec{x}, t) = -\bar{\psi}\gamma^i\psi(-\vec{x}, t). \end{aligned} \tag{76}$$

This implies that $\bar{\psi}\gamma^\mu\psi$ transforms as a vector, with the spatial part changing sign. Again, this is the transformation law under parity we expect from a vector. You can also check that $\bar{\psi}S^{\mu\nu}\psi$ transforms as a suitable tensor.

However, we can use γ^5 to form another Lorentz scalar and Lorentz vector:

$$\bar{\psi}\gamma^5\psi \quad \text{and} \quad \bar{\psi}\gamma^5\gamma^\mu\psi. \tag{77}$$

These have the appropriate transformation law as Lorentz scalar and Lorentz vector under continuous Lorentz transformation. However, under parity, these object transform like:

$$\begin{aligned} &P : \bar{\psi}\gamma^5\psi(\vec{x}, t) \to \bar{\psi}\gamma^0\gamma^5\gamma^0\psi(-\vec{x}, t) = -\bar{\psi}\gamma^5\psi(-\vec{x}, t) \\ &P : \bar{\psi}\gamma^5\gamma^\mu\psi(\vec{x}, t) \to \bar{\psi}\gamma^0\gamma^5\gamma^\mu\gamma^0\psi(-\vec{x}, t) = \begin{cases} -\bar{\psi}\gamma^5\gamma^0\psi(-\vec{x}, t) & \mu = 0 \\ +\bar{\psi}\gamma^5\gamma^i\psi(-\vec{x}, t) & \mu = i. \end{cases} \end{aligned} \tag{78}$$

Thus, both these objects obtain an extra minus (-) sign under parity transformation. We say that $\bar{\psi}\gamma^5\psi$ transforms as **pseudo-scalar**, while $\bar{\psi}\gamma^5\gamma^\mu\psi$ transforms as an **axial vector** (also known as **pseudo-vector**).

To summarize, we have the following spinor bilinears:

$$\begin{aligned} \bar{\psi}\psi : &\quad \text{scalar} \\ \bar{\psi}\gamma^\mu\psi : &\quad \text{vector} \\ \bar{\psi}S^{\mu\nu}\psi : &\quad \text{tensor} \\ \bar{\psi}\gamma^5\psi : &\quad \text{pseudoscalar} \\ \bar{\psi}\gamma^5\gamma^\mu\psi : &\quad \text{axial vector} \end{aligned} \tag{79}$$

The total number of bilinears is $1 + 4 + (4 \times 3/2) + 4 + 1 = 16$, which is all we could hope for from a 4-component object (Γ is a 4×4 matrix).

Armed now with new terms involving γ^5, we can start adding terms to our Lagrangian to construct new theories. Typically such terms will break parity invariance of the theory, although this is not always true. (For example, the term $\phi\bar{\psi}\gamma^5\psi$ doesn't break parity if ψ is itself a pseudoscalar).

Nature makes use of these parity violating interactions by using γ^5 in the weak force. A theory which treats $\psi_\pm$ on an equal footing is called a **vector-like theory**. A theory in which ψ_+ and ψ_- appear differently is called a **chiral theory**.

7. Majorana Fermions

Our spinor ψ^α is a complex object. It has to be because the representation $S[\Lambda]$ is typically also complex. This means that if we were to try to make ψ real, for example by imposing $\psi = \psi^\star$, then it wouldn't stay that way once we make a Lorentz transformation.

However, there is a way to impose a reality condition on the Dirac spinor ψ. To motivate this possibility, let us look at a novel basis for the Clifford algebra, known as the *Majorana basis*.

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \ , \ \gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix} \ , \ \gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix} \ , \ \gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix} . \qquad (80)$$

These matrices satisfy the Clifford algebra.

All these matrices have a special property: they are all pure imaginary, $(\gamma^\mu)^\star = -\gamma^\mu$. This means that the generators of the Lorentz group $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$, and hence the matrices $S[\Lambda]$ are all real. So with this basis of the Clifford algebra, we can work with a real spinor simply by imposing the condition,

$$\psi = \psi^\star \qquad (81)$$

which is preserved under Lorentz transformation. Such spinors are called **Majorana spinors**.

Let us now look at a general basis for the Clifford algebra. We only ask that the basis satisfies $(\gamma^0)^\dagger = \gamma^0$, and $(\gamma^i)^\dagger = -\gamma^i$. We define the **charge conjugate** of a Dirac spinor ψ as

$$\psi^{(c)} = C\psi^\star. \qquad (82)$$

Here, C is a 4×4 matrix satisfying

$$C^\dagger C = 1 \ \text{ and } \ C^\dagger \gamma^\mu C = -(\gamma^\mu)^\star. \qquad (83)$$

Let us see first that Equation 82 is a good definition, in the sense that $\psi^{(c)}$ transforms nicely under a Lorentz transformation. We have:

$$\psi^{(c)} \to CS[\Lambda]^\star \psi^\star = S[\Lambda]C\psi^\star = S[\Lambda]\psi^{(c)}, \tag{84}$$

where we used equation 83 in taking the matrix C through $S[\Lambda]^\star$.

In fact, not only does $\psi^{(c)}$ transform nicely under the Lorentz group, but if ψ satisfies the Dirac equation, then $\psi^{(c)}$ does too. This follows from

$$\begin{aligned}(i\partial\!\!\!/ - m)\psi = 0 \;\; &\Rightarrow \;\; (-i\partial\!\!\!/^\star - m)\psi^\star = 0 \\ &\Rightarrow \;\; C(-i\partial\!\!\!/^\star - m)\psi^\star = (+i\partial\!\!\!/ - m)\psi^{(c)} = 0.\end{aligned}$$

Finally, we can now impose the Lorentz invariant reality condition on the Dirac spinor, to yield a Majorana spinor,

$$\psi^{(c)} = \psi. \tag{85}$$

After quantization, the Majorana spinor gives rise to a fermion that is its own anti-particle. This is exactly the same as in the case of scalar fields, where we have seen that a real scalar field gives rise to a spin 0 boson that is its own anti-particle. (Note that in many texts an extra factor of γ^0 is absorbed into the definition of C).

So what is this matrix C? For a given representation of the Clifford algebra, it is something that we can find fairly easily. In the Majorana basis, where the gamma matrices are pure imaginary, we have simply $C_{\rm Maj} = \mathbf{1}$ and the Majorana condition $\psi = \psi^{(c)}$ becomes $\psi = \psi^\star$.

In the chiral basis (Equation 15) , only γ^2 is imaginary, and we may take $C_{\rm chiral} = i\gamma^2 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}$. (Note that the matrix $i\sigma^2$ that appears here is simply the anti-symmetric matrix $\epsilon^{\alpha\beta}$).

It is interesting to see how the Majorana condition (Equation 85) looks in terms of the decomposition into left and right handed Weyl spinors (Equation 55). Plugging in the various definitions, we find that $u_+ = i\sigma^2 u_-^\star$ and $u_- = -i\sigma^2 u_+^\star$. In other words, a Majorana spinor can be written in terms of Weyl spinors as

$$\psi = \begin{pmatrix} u_+ \\ -i\sigma^2 u_+^\star \end{pmatrix}. \tag{86}$$

Note that it is not possible to impose the Majorana condition $\psi = \psi^{(c)}$ at the same time as the Weyl condition ($u_- = 0$ or $u_+ = 0$). Instead the Majorana condition relates u_- and u_+.

An Aside: Spinors in Different Dimensions. The ability to impose Majorana or Weyl conditions on Dirac spinors depends on both the dimension and the signature of spacetime. One can always impose the Weyl condition on a spinor in even dimensional Minkowski space, basically because you can always build a suitable "γ^5" projection matrix by multiplying together all the other γ-matrices. The pattern for when the Majorana condition can be imposed is a little more sporadic. Interestingly, although the Majorana condition and Weyl condition cannot be imposed simultaneously in four dimensions, you can do this in Minkowski spacetimes of dimension $2, 10, 18, \ldots$

8. Symmetries and Conserved Currents

The Dirac Lagrangian enjoys a number of symmetries. Here we list them and compute the associated conserved currents.

Spacetime Translations

Under spacetime translations the spinor transforms as

$$\delta\psi = \epsilon^\mu \partial_\mu \psi \tag{87}$$

The Lagrangian depends on $\partial_\mu\psi$, but not $\partial_\mu\bar{\psi}$, so the standard formula ("Classical Fields", Eq. 41: $T^\mu{}_\nu = (\partial\mathcal{L}/\partial(\partial_\mu\phi))\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}$) gives us the energy-momentum tensor

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi - \eta^{\mu\nu}\mathcal{L}. \tag{88}$$

Since a current is conserved only when the equations of motion are obeyed, we can impose the equations of motion already on $T^{\mu\nu}$. In the case of a scalar field this didn't really buy us anything because the equations of motion are second order in derivatives, while the energy-momentum is typically first order. However, for a spinor field the equations of motion are first order: $(i\partial\!\!\!/ - m)\psi = 0$. This means we can set $\mathcal{L} = 0$ in $T^{\mu\nu}$, leaving

$$T^{\mu\nu} = i\bar{\psi}\gamma^\mu\partial^\nu\psi. \tag{89}$$

In particular, we have the total energy:

$$E = \int d^3x T^{00} = \int d^3x i\bar{\psi}\gamma^0\dot{\psi} = \int d^3x \psi^\dagger\gamma^0(-i\gamma^i\partial_i + m)\psi, \tag{90}$$

where, in the last equality, we have again used the equations of motion.

Lorentz Transformations

Under an infinitesimal Lorentz transformation, the Dirac spinor transforms as in Equation 21 which, in infinitesimal form, reads

$$\delta\psi^\alpha = -\omega^\mu{}_\nu x^\nu \partial_\mu \psi^\alpha + \frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta \psi^\beta. \tag{91}$$

Using Equation 9, we can write $\omega^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^\mu{}_\nu$, where the generators of the Lorentz algebra $\mathcal{M}^{\rho\sigma}$ are given in Equation 7,

$$(\mathcal{M}^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu.$$

Direct substitution thus yields $\omega^{\mu\nu} = \Omega^{\mu\nu}$. We thus get

$$\delta\psi^\alpha = -\omega^{\mu\nu}\left[x_\nu \partial_\mu \psi^\alpha - \frac{1}{2}(S_{\mu\nu})^\alpha{}_\beta \psi^\beta\right] \tag{92}$$

The conserved current arising from Lorentz transformations now follows from the same calculation we saw for the scalar field ("Classical Fields", Equation 54: $j^\mu = -\omega^\rho{}_\nu T^\mu{}_\rho x^\nu$) with two differences: firstly, as we saw above, the spinor equations of motion set $\mathcal{L} = 0$; secondly, we pick up an extra piece in the current from the second term in Equation 92. We have

$$(\mathcal{J}^\mu)^{\rho\sigma} = x^\sigma T^{\mu\rho} - x^\rho T^{\mu\sigma} - i\bar{\psi}\gamma^\mu S^{\rho\sigma}\psi \tag{93}$$

After quantization, when $(\mathcal{J}^\mu)^{\rho\sigma}$ is turned into an operator, this extra term will be responsible for providing the single particle states with internal angular momentum, telling us that the quantization of a Dirac spinor gives rise to a particle carrying spin 1/2.

Internal Vector Symmetry

The Dirac Lagrangian is invariant under rotating the phase of the spinor, $\psi \to e^{-i\alpha}\psi$. This gives rise to the current

$$j^\mu_V = \bar{\psi}\gamma^\mu\psi \tag{94}$$

(where "V" stands for "vector", reflecting the fact that the left and right-handed components $\psi_\pm$ transform in the same way under this symmetry).

We can easily check that j^μ_V is conserved under the equations of motion,

$$\partial_\mu j^\mu_V = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi) = im\bar{\psi}\psi - im\bar{\psi}\psi = 0. \tag{95}$$

Where, in the last equality, we have used the equations of motion $i\partial\!\!\!/\psi = m\psi$ and $i\partial_\mu\bar{\psi}\gamma^\mu = -m\bar{\psi}$.

The conserved quantity arising from this symmetry is

$$Q = \int d^3x \bar{\psi}\gamma^0\psi = \int d^3x \psi^\dagger\psi \tag{96}$$

We will see shortly that this has the interpretation of electric charge, or particle number, for fermions.

Axial Symmetry

When $m = 0$, the Dirac Lagrangian admits an extra internal symmetry which rotates left and right-handed fermions in opposite directions,

$$\psi \to e^{i\alpha\gamma^5}\psi \quad \text{and} \quad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma^5}. \tag{97}$$

(The second transformation follows the first after noting that $e^{-i\alpha\gamma^5}\gamma^0 = \gamma^0 e^{+i\alpha\gamma^5}$). This gives the conserved current,

$$j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \tag{98}$$

where A stands for "Axial", since j_A^μ is an axial vector. This is conserved only when $m = 0$. Indeed, with the full Dirac Lagrangian we may compute

$$\partial_\mu j_A^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi = 2im\bar{\psi}\gamma^5\psi, \tag{99}$$

which vanishes only for $m = 0$.

However, in the quantum theory things become more interesting for the axial current. When the theory is coupled to gauge fields (in a manner we will discuss later on), the axial transformation remains a symmetry of the classical Lagrangian. But it doesn't survive the quantization process. It is the archetypal example of an anomaly: a symmetry of the classical theory that is not preserved in the quantum theory.

9. Plane Wave Solutions to Dirac Equation

Let us now study solutions to the Dirac equation (48),

$$(i\gamma^\mu\partial_\mu - m)\psi = 0.$$

We start by making a simple ansatz:

$$\psi = u(\vec{p})e^{-ip\cdot x}, \tag{100}$$

where $u(\vec{p})$ is a four-component spinor, independent of spacetime x. As the notation suggests, it can depend on the 3-momentum $\vec{p}$. The Dirac equation then becomes

$$(\gamma^\mu p_\mu - m)u(\vec{p}) = \begin{pmatrix} -m & p_\mu\sigma^\mu \\ p_\mu\bar{\sigma}^\mu & -m \end{pmatrix} u(\vec{p}) = 0 \tag{101}$$

where again we use the definition,

$$\sigma^\mu = (1, \sigma^i) \quad \text{and} \quad \bar{\sigma}^\mu = (1, -\sigma^i). \tag{102}$$

Claim. The solution to Dirac Equation 101 (in the chiral representation) is

$$u(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\xi \\ \sqrt{p \cdot \bar{\sigma}}\xi \end{pmatrix}, \tag{103}$$

for any 2-component spinor ξ which we will normalize to $\xi^\dagger \xi = 1$.

Proof. Let us write $u(\vec{p})^T = (u_1, u_2)$. Then Equation 101 reads

$$(p \cdot \sigma)u_2 = mu_1 \quad \text{and} \quad (p \cdot \bar{\sigma})u_1 = mu_2 \tag{104}$$

Either one of these equations implies the other, a fact which follows from the identity $(p \cdot \sigma)(p \cdot \bar{\sigma}) = p_0^2 - p_i p_j \sigma^i \sigma^j = p_0^2 - p_i p_j \delta^{ij} = p_\mu p^\mu = m^2$.

Thus, let's try the ansatz $u_1 = (p \cdot \sigma)\xi'$ for some spinor ξ'. The second Equation in 104 implies $u_2 = m\xi'$. So we learn that any spinor of the form

$$u(\vec{p}) = A \begin{pmatrix} (p \cdot \sigma)\xi' \\ m\xi' \end{pmatrix} \tag{105}$$

with some constant A is a solution to Equation 101. To make this more symmetric, we choose $A = 1/m$ and $\xi' = \sqrt{p \cdot \bar{\sigma}}\xi$, with constant ξ. Then we obtain $u_1 = (p \cdot \sigma)\sqrt{p \cdot \bar{\sigma}}\xi = m\sqrt{p \cdot \sigma}\xi$, which leads to result in Equation 103.

Negative Frequency Solutions

We can get further solutions to the Dirac Equation using the ansatz

$$\psi = v(\vec{p})e^{+ip \cdot x}. \tag{106}$$

Solutions of the form given by Equation 100, which oscillate in time as $\psi \sim e^{-iEt}$ are called *positive frequency* solutions. Those of the form given by equation 106, which oscillates as $\psi \sim e^{+iEt}$, are *negative frequency* solutions. Note though, that both are solutions to the classical field equations, and both have positive energy (see Equation 90). The Dirac equation requires that the 4-component spinor $v(\vec{p})$ satisfies

$$(\gamma^\mu p_\mu + m)v(\vec{p}) = \begin{pmatrix} m & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & m \end{pmatrix} v(\vec{p}) = 0, \tag{107}$$

which is solved by

$$v(\vec{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma}\eta \\ -\sqrt{p \cdot \bar{\sigma}}\eta \end{pmatrix}, \tag{108}$$

for some 2-component spinor η, which we take to be constant and normalized to $\eta^\dagger \eta = 1$.

9.1. Examples

Consider the positive frequency solution to Dirac's equation with mass m and 3-momentum $\vec{p} = 0$,

$$u(\vec{p}) = \sqrt{m}\begin{pmatrix}\xi \\ \xi\end{pmatrix}, \tag{109}$$

where ξ is any 2-component spinor. Spatial rotations of the field act on ξ by Equation 25,

$$\xi \to e^{+i\vec{\varphi}\cdot\vec{\sigma}/2}\xi. \tag{110}$$

The 2-component spinor ξ defines the **spin** of the field.

This should be familiar from quantum mechanics. A field with spin up (down) along a given direction is described by the eigenvector of the corresponding Pauli matrix with eigenvalue +1 (-1 respectively). For example, $\xi^T = (1, 0)$ describes a field with spin up along the z-axis. After quantization, this will become the spin of the associated particle. In the rest of this section, we will indulge in an abuse of terminology and refer to the classical solutions to the Dirac equations as "particles", even though they have no such interpretation before quantization.

Consider now boosting the particle with spin $\xi^T = (1, 0)$ along the x^3 direction, with $p^\mu = (E, 0, 0, p)$. The solution to the Dirac equation becomes

$$u(\vec{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}\begin{pmatrix}1\\0\end{pmatrix} \\ \sqrt{p\cdot\bar{\sigma}}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\sqrt{E-p^3}\begin{pmatrix}1\\0\end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \tag{111}$$

In fact, this expression also makes sense for a massless field, for which $E = p^3$. (We picked the normalization in Equation 103 for the solutions so that this would be the case).

For a massless particle we have

$$u(\vec{p}) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}. \tag{112}$$

Similarly, for a boosted solution of the spin down $\xi^T = (0, 1)$ field, we have

$$u(\vec{p}) = \begin{pmatrix}\sqrt{p\cdot\sigma}\begin{pmatrix}0\\1\end{pmatrix} \\ \sqrt{p\cdot\bar{\sigma}}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}\sqrt{E+p^3}\begin{pmatrix}0\\1\end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} \overset{m\to 0}{\to} \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \tag{113}$$

Helicity
The helicity operator is defined to be the projection of the angular momentum along the direction of momentum:

$$h = \frac{i}{2}\epsilon_{ijk}\hat{p}^i S^{jk} = \frac{1}{2}\hat{p}_i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \tag{114}$$

where S^{ij} is the rotation generator given in Equation 24. The massless field with spin $\xi^T = (1,0)$ in Equation 112 has helicity $h = 1/2$; we say that it is **right-handed**. Similarly, the field in Equation 113 has helicity $h = -1/2$, and is called **left-handed**.

9.2. Some Useful Formulae: Inner and Outer Products

There are a number of identities that will be very useful in the following, regarding the inner (and outer) products of the spinors $u(\vec{p})$ and $v(\vec{p})$.

In deriving these identities, it is convenient to introduce a basis ξ^s and η^s, with $s = 1,2$ for the two-component spinors such that

$$\xi^{r\dagger}\xi^s = \delta^{rs} \quad \text{and} \quad \eta^{r\dagger}\eta^s = \delta^{rs}. \tag{115}$$

For example, we can take

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{116}$$

and similarly for η^s.

Let us look first at the positive frequency plane waves. We write the two independent solutions as

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \tag{117}$$

We can take the inner product of four-component spinors in two different ways: either as $u^\dagger \cdot u$, or as $\bar{u}\cdot u$. Of course, only the latter will be Lorentz invariant, but it turns out that the former is needed when we come to quantize the theory. Here we state both:

$$\begin{aligned} u^{r\dagger}(\vec{p})\cdot u^s(\vec{p}) &= \left(\xi^{r\dagger}\sqrt{p\cdot\sigma}, \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\right)\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \\ &= \xi^{r\dagger}p\cdot\sigma\xi^s + \xi^{r\dagger}p\cdot\bar{\sigma}\xi^s \\ &= 2\xi^{r\dagger}p_0\xi^s \\ &= 2p_0\delta^{rs} \end{aligned} \tag{118}$$

while the Lorentz invariant inner product is

$$\bar{u}^r(\vec{p}) \cdot u^s(\vec{p}) = \left(\xi^{r\dagger}\sqrt{p\cdot\sigma}, \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\right)\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} = 2m\delta^{rs} \tag{119}$$

We have analogous results for the negative frequency solutions, which we may write as

$$v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix}, \tag{120}$$

with $v^{r\dagger}(\vec{p}) \cdot v^s(\vec{p}) = 2p_0\delta^{rs}$, and similarly $\bar{v}^r(\vec{p}) \cdot v^s(\vec{p}) = -2m\delta^{rs}$.

We can also compute the inner product between u and v. We have

$$\begin{aligned} \bar{u}^r(\vec{p}) \cdot v^s(\vec{p}) &= \left(\xi^{r\dagger}\sqrt{p\cdot\sigma}, \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\right)\gamma^0\begin{pmatrix} \sqrt{p\cdot\sigma}\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\eta^s \end{pmatrix} \\ &= \xi^{r\dagger}\sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)}\eta^s - \xi^{r\dagger}\sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)}\eta^s = 0 \end{aligned} \tag{121}$$

and similarly, $\bar{v}^r(\vec{p}) \cdot u^s(\vec{p}) = 0$.

However, when we come to $u^\dagger \cdot v$, it is a slightly different combination that has nice properties (and this same combination appears when we quantize the theory). We look at $u^{r\dagger}(\vec{p}) \cdot v^s(-\vec{p})$, with the 3-momentum in the spinor v taking the opposite sign. Defining the 4-momentum $(p')^\mu = (p^0, -\vec{p})$, we have

$$\begin{aligned} u^{r\dagger}(\vec{p}) \cdot v^s(-\vec{p}) &= \left(\xi^{r\dagger}\sqrt{p\cdot\sigma}, \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\right)\begin{pmatrix} \sqrt{p'\cdot\sigma}\eta^s \\ -\sqrt{p'\cdot\bar{\sigma}}\eta^s \end{pmatrix} \\ &= \xi^{r\dagger}\sqrt{(p\cdot\sigma)(p'\cdot\sigma)}\eta^s - \xi^{r\dagger}\sqrt{(p\cdot\bar{\sigma})(p'\cdot\bar{\sigma})}\eta^s \end{aligned} \tag{122}$$

Now the terms under the square-root are given by $(p\cdot\sigma)(p'\cdot\sigma) = (p_0 + p_i\sigma^i)(p_0 - p_i\sigma^i) = p_0^2 - \vec{p}^2 = m^2$. The same expression holds for $(p\cdot\bar{\sigma})(p'\cdot\bar{\sigma})$, and thus the two terms cancel. We thus find that

$$u^{r\dagger}(\vec{p}) \cdot v^s(-\vec{p}) = v^{r\dagger}(\vec{p}) \cdot u^s(-\vec{p}) = 0 \tag{123}$$

Outer products

The last spinor identity that we need before going into quantum theory is

$$\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p} + m, \tag{124}$$

where the two spinors are *not* contracted, but instead placed back to back to give a 4×4 matrix. Also,

$$\sum_{s=1}^{2} v^s(\vec{p})\bar{v}^s(\vec{p}) = \not{p} - m. \tag{125}$$

Proof.

$$\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \sum_{s=1}^{2} \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}}, \xi^{s\dagger}\sqrt{p\cdot\sigma}\right). \tag{126}$$

We can now use $\sum_s \xi^s \xi^{s\dagger} = \mathbf{1}$ (a 2×2 unit matrix), and get

$$\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} \tag{127}$$

which is the desired result.

A similar proof holds for $\sum_{s=1}^{2} v^s(\vec{p})\bar{v}^s(\vec{p})$.

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