Einstein's field equation

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This part of the course is based on Refs. [1], [2] and [3].

1. Derivation of the Equation

Finally, we have all the tools needed to work out Einstein's field Equation, which explains how the metric responds to energy and momentum. The basic idea is that existence of energy (which is equal to mass, according to $E = mc^2$) curves space time. In deriving this equation we will use somewhat informal arguments, which are in fact close to the way Einstein himself was thinking, as this is (to my opinion) the most straightforward way. For those interested, Carroll provides a second derivation starting from the action and deriving the corresponding Equation of motion.

We begin with the realization that we would like to find an equation which supersedes the Poisson equation for the Newtonian potential:

$$\nabla^2 \Phi = 4\pi G \rho \ , \tag{1}$$

where $\nabla^2 = \delta^{ij}\partial_i\partial_j$ is the Laplacian in space and ρ is the mass density. (The explicit form of $\Phi = -GM/r$ is one solution of Eq. 1, for the case of a point-like mass distribution.)

What characteristics should our sought-after equation possess? The logic goes as follows. (I) On the left-hand side of Equation 1 we have a second-order differential operator acting on the gravitational potential, and on the right-hand side a measure of the mass distribution. (II) A relativistic generalization should take the form of an equation between tensors.

The tensor generalization of the mass density is the energy-momentum tensor $T_{\mu\nu}$. The gravitational potential, meanwhile, should get replaced by the metric tensor. It is thus reasonable to guess that the new equation will have $T_{\mu\nu}$ set proportional to some tensor which is second-order in derivatives of the metric. In fact, using the Newtonian limit for the metric $g_{00} = -(1 + 2\Phi)$ and $T_{00} = \rho$, we see that in this limit we are looking for an equation that predicts

$$\nabla^2 h_{00} = -8\pi G T_{00} \ , \tag{2}$$

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with $h_{00} \equiv 2\Phi$. We do though need to generalize it to a completely tensorial equation.

The left-hand side of Eq. 2 does not obviously generalize to a tensor. The first choice might be to act the D'Alembertian $\Box = \nabla^\mu \nabla_\mu$ on the metric $g_{\mu\nu}$, but this is automatically zero by metric compatibility ($\equiv g_{\mu\nu;\lambda} = 0$).

Fortunately, there is an obvious quantity which is not zero and is constructed from second derivatives (and first derivatives) of the metric: the Riemann tensor $R^\rho{}_{\sigma\mu\nu}$. It doesn't have the right number of indices, but we can contract it to form the Ricci tensor $R_{\mu\nu}$, which does; furthermore, it is symmetric. It is therefore reasonable to guess that the gravitational field equations are

$$R_{\mu\nu} = \kappa T_{\mu\nu} \ , \tag{3}$$

for some constant κ. In fact, Einstein did suggest this equation at one point.

Unfortunately, this suggestion too doesn't work, as there is a problem with energy conservation. According to the Principle of Equivalence, the statement of energy-momentum conservation in curved spacetime should be

$$\nabla^\mu T_{\mu\nu} = 0 \ , \tag{4}$$

which would then imply

$$\nabla^\mu R_{\mu\nu} = 0 \ . \tag{5}$$

This is certainly not true in an arbitrary geometry; recall that when we discussed Bianchi identity, we got

$$\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R \ . \tag{6}$$

But our proposed field equation implies that $R = \kappa g^{\mu\nu} T_{\mu\nu} = \kappa T$, so taking these together we have

$$\nabla_\mu T = 0 \ . \tag{7}$$

The covariant derivative of a scalar is just the partial derivative, so Equation 7 is telling us that T is constant throughout spacetime. This is highly implausible, since $T = 0$ in vacuum while $T > 0$ in matter. We have to try harder.

(Actually we are cheating slightly, in taking the equation $\nabla^\mu T_{\mu\nu} = 0$ so seriously. If as we said, the equivalence principle is only an approximate guide, we could imagine that there are nonzero terms on the right-hand side involving the curvature tensor. Later we will be more precise and argue that they are strictly zero.)

By now we are quiet close. We already know of a symmetric $(0, 2)$ tensor, constructed from the Ricci tensor, which is automatically conserved: the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \ , \tag{8}$$

which always obeys $\nabla^\mu G_{\mu\nu} = 0$. We are therefore led to propose

$$G_{\mu\nu} = \kappa T_{\mu\nu} \tag{9}$$

as a field equation for the metric. This equation satisfies all of the obvious requirements; the right-hand side is a covariant expression of the energy and momentum density in the form of a symmetric and conserved $(0,2)$ tensor, while the left-hand side is a symmetric and conserved $(0,2)$ tensor constructed from the metric and its first and second derivatives. Equation 9 looks very promising; it only remains to see whether it actually reproduces gravity as we know it.

To answer this, note that contracting both sides of Equation 9 yields (in four dimensions)

$$g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}) = R - \frac{1}{2}Rg^{\mu\nu}g_{\mu\nu} = \kappa T \to R = -\kappa T \ , \tag{10}$$

and using this we can rewrite Equation 9 as

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}) \ . \tag{11}$$

This is identical to equation 9, just written slightly differently.

Let us show now that this equation predicts Newtonian gravity in the weak-field, time-independent, slowly-moving-particles limit. In this limit the rest energy $\rho = T_{00}$ will be much larger than the other terms in $T_{\mu\nu}$, so we focus on the $\mu = 0$, $\nu = 0$ component of Equation 11. Recall that in the weak-field limit we have $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, with $|h_{\mu\nu}| \ll 1$, and $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. We can thus write

$$\begin{aligned} g_{00} &= -1 + h_{00} \ , \\ g^{00} &= -1 - h_{00} \ . \end{aligned} \tag{12}$$

The trace of the energy-momentum tensor, to lowest nontrivial order, is

$$T = g^{00}T_{00} = -T_{00} \ . \tag{13}$$

Plugging this into Equation 11, we get

$$R_{00} = \frac{1}{2}\kappa T_{00} \ . \tag{14}$$

This is an equation relating derivatives of the metric to the energy density. To find the explicit expression in terms of the metric (rather than its derivatives), we need to evaluate $R_{00} = R^\lambda{}_{0\lambda0}$. In fact we only need $R^i{}_{0i0}$, since $R^0{}_{000} = 0$. We have

$$R^i{}_{0j0} = \partial_j\Gamma^i_{00} - \partial_0\Gamma^i_{j0} + \Gamma^i_{j\lambda}\Gamma^\lambda_{00} - \Gamma^i_{0\lambda}\Gamma^\lambda_{j0} \ . \tag{15}$$

The second term here is a time derivative, which vanishes for static fields. The third and fourth terms are of the form $(\Gamma)^2$, and since Γ is first-order in the metric perturbation these contribute only at second order, and can be neglected. Thus, to first order we are left with only the first term, $R^i{}_{0j0} = \partial_j \Gamma^i_{00}$. From this we get

$$\begin{aligned} R_{00} &= R^i{}_{0i0} \\ &= \partial_i \left(\tfrac{1}{2} g^{i\lambda}(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00})\right) \\ &= -\tfrac{1}{2}\eta^{ij}\partial_i\partial_j h_{00} \\ &= -\tfrac{1}{2}\nabla^2 h_{00} \ . \end{aligned} \tag{16}$$

Comparing to Equation 14, we see that in the Newtonian limit, the 00 component of Equation 9 becomes

$$\nabla^2 h_{00} = -\kappa T_{00} \ . \tag{17}$$

But this is identical to Equation 2, if we set $\kappa = 8\pi G$.

We have thus shown that in the Newtonian limit, Equation 9 indeed retrieves the familiar Newtonian result gravitational potential. This guess thus seem to be a good one. With the normalization fixed by comparison with the Newtonian limit, we can present **Einstein's field equations** for general relativity:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \ . \tag{18}$$

These tell us how the curvature of spacetime reacts to the presence of energy-momentum. Einstein, you may have heard, thought that the left-hand side was nice and geometrical, while the right-hand side was somewhat less compelling.

Einstein's Equation is the most fundamental equation of general relativity. The way we introduced it here is as a generalization of Poisson's equation for the Newtonian gravitational potential. **Its importance is that it expresses how the presence of energy (mass) source curves space time.** Pretty much what we are going to do from now until the end of the course is to explore its consequences, and look for solutions for this equation. The description will be split into two parts: in the first part, we will explore vacuum solutions ($T_{\mu\nu} = 0$): in this category falls most of what we discussed so far, such as astronauts (or other objects) moving in space in the presence of external gravitation field. In fact, you already know one solution to the equation - this is the flat Minkowski metric. However, clearly there are other solutions, the second most important one (after the Minkowski metric) is the Schwarzschield solution.

In the last part of the semester, we will write Einstein's equation for the entire universe. Clearly, the universe is not empty, and hence the right hand side of Equation 18 is non-zero.

We will explore solutions in this case, the most important one results in the (Friedman)-Robertson-Walker (FRW) metric that describes the evolution of the universe as a whole. This branch of physics is known as Cosmology.

1.1. On the complexity of Einstein's equation

Einstein's equations may be thought of as second-order differential equations for the metric tensor field $g_{\mu\nu}$. There are *ten independent equations* (since both sides are symmetric two-index tensors), which seems to be exactly right for the ten unknown functions of the metric components. However, the Bianchi identity $\nabla^{\mu} G_{\mu\nu} = 0$ represents four constraints on the functions $R_{\mu\nu}$, so there are only six truly independent equations in Equation 18. In fact this is appropriate, since if a metric is a solution to Einstein's equation in one coordinate system x^{μ} it should also be a solution in any other coordinate system $x^{\mu'}$. This means that there are four unphysical degrees of freedom in $g_{\mu\nu}$ (represented by the four functions $x^{\mu'}(x^{\mu})$), and we should expect that Einstein's equations only constrain the six coordinate-independent degrees of freedom.

As differential equations, these are extremely complicated; the Ricci scalar and tensor are contractions of the Riemann tensor, which involves derivatives and products of the Christoffel symbols, which in turn involve the inverse metric and derivatives of the metric. Furthermore, the energy-momentum tensor $T_{\mu\nu}$ will generally involve the metric as well. The equations are also nonlinear, so that two known solutions cannot be superposed to find a third. It is therefore very difficult to solve Einstein's equations in any sort of generality, and it is usually necessary to make some simplifying assumptions. Even in vacuum, where we set the energy-momentum tensor to zero, the resulting equations (using Equation 11)

$$R_{\mu\nu} = 0 \tag{19}$$

can be very difficult to solve. Thus, in order to actually solve Einstein's equation, often people use the simplifying assumption that the metric has a significant degree of symmetry. This of course simplifies considerably the equation.

2. The cosmological constant

You may have noticed that there is another extra term that could be added to the left hand side of Einstein's field equation (Eq. 18), consistent with local conservation of $T_{\mu\nu}$. This is a term of the form $\Lambda g_{\mu\nu}$, for some constant Λ. Adding it to the left hand side does not affect local conservation, because the covariant derivative of the metric is zero. The term Λ is called the **cosmological constant**. The resulting field equation (in vacuum) is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \ , \tag{20}$$

Einstein's original motivation for introducing Λ was that it became clear that there were no solutions to his equations representing a static cosmology (a universe unchanging with time on large scales) with a nonzero matter content. Indeed, it was believed in that times that the universe is static. If the cosmological constant is tuned just right, it is possible to find a static solution, but it is unstable to small perturbations.

This changed a few years later, when Hubble proved that the universe is in fact expanding, hence it is **not** static (which Einstein's equation would predict if the cosmological constant was not added; to this Einstein referred to as the biggest mistake of his life). This discovery led Einstein to reject his own suggestion.

The cosmological constant, though, made a great re-appearance. In modern day, the $\Lambda g_{\mu\nu}$ term is moved to the right hand side, and one can think of it as a kind of energy-momentum tensor, with $T_{\mu\nu} = -\Lambda g_{\mu\nu}$ (it is automatically conserved by metric compatibility). Then Λ can be interpreted as the "energy density of the vacuum," a source of energy and momentum that is present even in the absence of matter fields. This interpretation is important because quantum field theory predicts that the vacuum should have some sort of energy and momentum. In ordinary quantum mechanics, an harmonic oscillator with frequency ω and minimum classical energy $E_0 = 0$ upon quantization has a ground state with energy $E_0 = \frac{1}{2}\hbar\omega$. A quantized field can be thought of as a collection of an infinite number of harmonic oscillators, and each mode contributes to the ground state energy. The result is of course infinite, and must be appropriately regularized, for example by introducing a cutoff at high frequencies. The final vacuum energy, which is the regularized sum of the energies of the ground state oscillations of all the fields of the theory, has no good reason to be zero and in fact would be expected to have a natural scale

$$\Lambda \sim m_P^4 \ , \tag{21}$$

where the Planck mass m_P is approximately 10^{19} GeV, or 10^{-5} grams. Observations of the universe on large scales allow us to constrain the actual value of Λ, which turns out to be smaller than the prediction of Equation 21 by at least a factor of 10^{120}(!). This is the largest known discrepancy between theoretical estimate and observational constraint in physics, and convinces many people that the "cosmological constant problem" is one of the most important unsolved problems today. On the other hand the observations do not tell us that Λ is strictly zero (on the contrary, it isn't), and in fact allow values that can have important consequences for the evolution of the universe. This mistake of Einstein's therefore continues to bedevil both physicists and astronomers. (Further discussion is found in QFT course).

3. Symmetries and Killing vectors

In order to search for solutions to Einstein's equation, our best bet is thus to try and use metric which has symmetric properties. The main problem though can be phrased something along: "we would like to use the symmetry of the metric space in order to get information about the metric, but how can we do that before we know the metric which tells us the symmetry ?". Thus, what we really need is a way to describe a symmetry in a *covariant* language, namely, independent on particular coordinate system. This is done by means of **Killing vectors**.[2]

We say that a manifold, $\mathcal{M}$ possess a symmetry if the geometry is invariant under a certain transformation that maps $\mathcal{M}$ into itself. In other words, the metric is the same in two different points on $\mathcal{M}$. A symmetry of the metric is called **isometry**.

As a simple example, we may look at Minkowski space,

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2.$$

There are several isometries of this space, such as translations ($x^\mu \to x^\mu + a^\mu$), or Lorentz transformations ($x^\mu \to \Lambda^\mu{}_\nu x^\nu$).

Indeed, we can immediately recognize that the metric is invariant under translations, as the metric coefficients $\eta_{\mu\nu}$ are independent of the individual coordinate functions, x^μ. This is a generally true statement: whenever $\partial_\sigma g_{\mu\nu} = 0$ for some fixed direction σ (but any μ, ν), there is a symmetry under translation along x^σ:

$$\partial_\sigma g_{\mu\nu} = 0 \quad \to \quad x^\sigma \to x^\sigma + a^\sigma \text{ is a symmetry.} \tag{22}$$

3.1. Isometries and the motion of test particles

Isometries are particularly important when considering the motion of test particles which move along geodesics: the momentum component, p_σ is a conserved quantity of the motion.

[2] After the mathematician Wilhelm Killing, not because it is particularly difficult!.

The proof goes as follows. Recall that we showed (at the end of the chapter about "Tensors"), that the geodesic equation (at least, for a time-like geodesic) can be written as

$$\nabla_U U = U^\lambda \nabla_\lambda U^\mu = 0, \tag{23}$$

where $U^\mu = dx^\mu/d\tau$ is the tangent vector along the path, and can therefore be rightfully considered as the 4-velocity.

Similarly, we can write this in terms of the 4-momentum $p^\mu = mU^\mu$,

$$p^\lambda \nabla_\lambda p^\mu = 0 \tag{24}$$

Using the metric compatibility $\nabla_\lambda g_{\mu\rho} = 0$ we are free to lower the index μ. We expand the covariant derivative to obtain

$$p^\lambda \partial_\lambda p_\mu - \Gamma^\rho_{\lambda\mu} p^\lambda p_\rho = 0 \tag{25}$$

The first term expresses the change of the momentum components along the path:

$$p^\lambda \partial_\lambda p_\mu = m \frac{dx^\lambda}{d\tau} \partial_\lambda p_\mu = m \frac{dp_\mu}{d\tau} \tag{26}$$

The second term gives

$$\begin{aligned} \Gamma^\rho_{\lambda\mu} p^\lambda p_\rho &= \tfrac{1}{2} g^{\rho\nu} \left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}\right) p^\lambda p_\rho \\ &= \tfrac{1}{2} \left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}\right) p^\lambda p^\nu \\ &= \tfrac{1}{2} \left(\partial_\mu g_{\nu\lambda}\right) p^\lambda p^\nu, \end{aligned} \tag{27}$$

where in the last line we used the symmetry of $p^\lambda p^\nu$ to cancel the first and third terms on the right.

We thus find that the geodesic equation takes the form:

$$m \frac{dp_\mu}{d\tau} = \frac{1}{2} \left(\partial_\mu g_{\nu\lambda}\right) p^\lambda p^\nu. \tag{28}$$

Note that this calculation is completely general, and so far we made no assumption about any symmetry.

However, now we see that if all the metric coefficients are independent of the coordinate x^σ, namely $\partial_\sigma g_{\mu\nu} = 0$, we find that this isometry implies that the momentum component p_σ is a conserved quantity of the motion,

$$\partial_\sigma g_{\mu\nu} = 0 \quad \rightarrow \quad \frac{dp_\sigma}{d\tau} = 0. \tag{29}$$

This result, in fact, holds along any geodesic - though you might noted that we derived it only for time-like geodesic.

The fact that such conserved quantities exist implies that isometries are extremely useful in studying the motion of test particles in curved manifolds.

3.2. General description of symmetries: Killing vectors

Independence of the metric components on one or more coordinates implies the existence of an isometry. The converse, though, does not necessarily hold: for example, in Minkowski space there are 4 translations and 6 Lorentz transformations - total of 10, which is larger than the dimension of the space (4). Thus, the existence of an isometry does not always manifest itself in a simple way.

In order to develop a systematic way of finding isometries, we proceed as follows. We promote the right hand side of Equation 29, which represents the conservation of one component of the momentum to a covariant form. Assume that $g_{\mu\nu}$ is independent of the coordinate x^σ. We consider the vector

$$K \equiv \partial_{\sigma_\star}, \tag{30}$$

or, in component notation

$$K^\mu = (\partial_{\sigma_\star})^\mu = \delta^\mu_{\sigma_\star}. \tag{31}$$

We say that the vector K^μ **generates** the isometry; that is we express an infinitesimal transformation that leaves the geometry invariant as a transformation along K^μ.

Using this vector, we write each component of the momentum (a scalar) in a manifestly covariant form:

$$p_{\sigma_\star} = K^\mu p_\mu = K_\mu p^\mu \tag{32}$$

(note the somewhat misleading notation - $p_{\sigma_\star}$ represents a single component of the momentum, hence the $\sigma_\star$).

Next, we note that the conservation of this momentum component along the geodesic, is equivalent to the statement that its directional derivative along the geodesic vanishes. Namely,

$$\frac{dp_{\sigma_\star}}{d\tau} = 0 \quad \Leftrightarrow \quad p^\mu \nabla_\mu (K_\nu p^\nu) = 0. \tag{33}$$

Expanding the expression on the right, we get

$$\begin{aligned} p^\mu \nabla_\mu (K_\nu p^\nu) &= p^\mu K_\nu \nabla_\mu p^\nu + p^\mu p^\nu \nabla_\mu K_\nu \\ &= p^\mu p^\nu \nabla_\mu K_\nu \\ &= \tfrac{1}{2} p^\mu p^\nu \left(\nabla_\mu K_\nu + \nabla_\nu K_\mu \right). \end{aligned} \tag{34}$$

Here, in the second line we used the geodesic Equation ($p^\mu \nabla_\mu p^\nu = 0$) to remove the first term on the right. In the third line, we used the fact that $p^\mu p^\nu$ is automatically symmetric in μ and ν, so only the symmetric part of $\nabla_\mu K_\nu$ has a non-zero contribution.

We therefore conclude that for any vector K^μ that satisfies

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu \equiv K_{\nu;\mu} + K_{\mu;\nu} = 0 \tag{35}$$

implies that $K_\nu p^\nu$ is conserved along a geodesic trajectory:

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \quad \Rightarrow \quad p^\mu \nabla_\mu (K_\nu p^\nu) = 0. \tag{36}$$

Equation 35 is known as **Killing Equation**. Any four-vector $K^\mu(x)$ that satisfies this Equation is said to be a **Killing vector field**, or simply **Killing vector** of the metric $g_{\mu\nu}(x)$.

Clearly, if the metric is manifestly independent of some coordinate x^σ, the vector ∂_σ will satisfy Killing's equation. However, Killing's equation is more general: if a vector K^μ satisfies Killing's Equation, it is always possible to find a coordinate system in which $K = \partial_\sigma$. This can be seen by recalling that we derived the Killing Equation 35 using this same requirement (see Equation 29). I provide a second proof of that, which is based on the derivative in Weinberg, in the appendix.

As a final remark, note that in $n \geq 2$ dimensions, there can be more Killing vectors than dimensions. This is because a set of Killing vector fields can be linearly independent, although at any one point on the manifold the vectors at that point are linearly dependent. This is because in general, the coefficients in the linear combination of the Killing vector fields may not be constant, but vary over the manifold.

A. A second, direct derivative of Killing Equation

Weinberg provides a different, more direct (I think) way of deriving Killing Equation, which is somewhat more subtle; still, I think it is useful to give it as a reference, as in some basic sense it complements the derivative presented above.

Consider a general manifold, $\mathcal{M}$, and a vector field $V^\mu(x)$ defined at the vicinity of the point x on a manifold. We define the **integral curves** of the vector field to be those curves $x^\mu(t)$ which solve

$$\frac{dx^\mu}{dt} = V^\mu \ . \tag{A1}$$

Note that this familiar-looking equation is now to be interpreted in the opposite sense from our usual way — we are given the vectors, from which we define the curves. Solutions to Equation A1 are guaranteed to exist as long as we don't do anything silly like run into the edge of our manifold; any standard differential geometry text will have the proof, which amounts to finding a clever coordinate system in which the problem reduces to the fundamental theorem of ordinary differential equations.

The vector V^μ thus defines a curve on the manifold which we parameterized by t. We can now change the coordinates along this curve from x^μ to $x^\mu + t$; while this can be thought of in the usual way of coordinate change, what we mean is that we are defining a map $\phi : M \to M$ which "move the points on the manifold, and then evaluate the coordinates of the new points" (this is a specific example of a *diffeomorphism*, which is an invertible, smooth function that maps a manifold to another).

Consider now a tensor T defined over all space. We say that the tensor is **form-invariant**, or simply **invariant** under coordinate transformation $x^{\mu'} = x^\mu + tV^\mu$ ($|t| \ll 1$) if $T(x^\mu) = T(x^{\mu'})$. The transformation is called **symmetry**.

The tensor that we are interested in is the metric tensor. Recall that under general coordinate transformation, *at any given point* x^μ the metric tensor transforms as

$$g_{\mu'\nu'}(x') = \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} g_{\alpha\beta}(x), \tag{A2}$$

or alternatively

$$g_{\mu\nu}(x) = \frac{\partial x^{\alpha'}}{\partial x^{\mu}} \frac{\partial x^{\beta'}}{\partial x^{\nu}} g_{\alpha'\beta'}(x'). \tag{A3}$$

We now make use of a very delicate point: x and x' corrspond to *the same physical point* that is expressed in different coordinate systems. In the different frames (unprimed and primed) it is expressed as *two different coordinates*. (Think of a translation: e.g., $x' = x + 3$. The point $x = 0$ and $x' = 3$ correspond to the same physical point, but in the different frames it is described by different coordinates).

We now use the assumption that $g_{\alpha'\beta'}$ is form invariant under the transformation $x^{\mu'} = x^\mu + tV^\mu$, namely when moving from x^μ to $x^{\mu'}$, the metric does not change: $g_{\alpha'\beta'}(x') = g_{\alpha\beta}(x) = g_{\alpha\beta}(x')$ to write

$$g_{\mu\nu}(x) = \frac{\partial x^{\alpha'}}{\partial x^{\mu}} \frac{\partial x^{\beta'}}{\partial x^{\nu}} g_{\alpha\beta}(x + tV) \tag{A4}$$

A transformation that fulfills Equation A4 (namely, for which the metric tensor is form-invariant) is called **isometry**, which we encountered earlier. The vector field $V^\mu(x)$ which fulfills this condition is the **Killing vector field**.

The condition that V^μ be a Killing vector field can be found by using the assumption $|t| \ll 1$ and writing Equation A4 to first order in t,

$$\begin{aligned} g_{\mu\nu} &= \left(\delta^\alpha_\mu + t\frac{\partial V^\alpha}{\partial x^\mu}\right)\left(\delta^\beta_\nu + t\frac{\partial V^\beta}{\partial x^\nu}\right)\left(g_{\alpha\beta} + \frac{\partial g_{\alpha\beta}}{\partial x^\kappa} tV^\kappa\right) \\ 0 &= \frac{\partial V^\alpha}{\partial x^\mu} g_{\alpha\nu} + \frac{\partial V^\beta}{\partial x^\nu} g_{\mu\beta} + \frac{\partial g_{\mu\nu}}{\partial x^\kappa} V^\kappa. \end{aligned} \tag{A5}$$

Using $V_\sigma = g_{\mu\sigma} V^\mu$ and taking the differential of $(V^\alpha g_{\alpha\nu})$ etc., we can write Equation A5 as

$$\begin{aligned} 0 &= \frac{\partial V_\nu}{\partial x^\mu} + \frac{\partial V_\mu}{\partial x^\nu} + V^\kappa \left(\frac{\partial g_{\mu\nu}}{\partial x^\kappa} - \frac{\partial g_{\kappa\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\kappa}}{\partial x^\nu}\right) \\ &= \frac{\partial V_\nu}{\partial x^\mu} + \frac{\partial V_\mu}{\partial x^\nu} - 2V_\kappa \Gamma^\kappa_{\mu\nu} \end{aligned} \tag{A6}$$

or, in a compact form,

$$V_{\mu;\nu} + V_{\nu;\mu} = 0, \tag{A7}$$

which is the familiar Killing Equation, 35. Any four-vector $V_\mu(x)$ that satisfies Equation A7 is said to be a **Killing vector** of the metric $g_{\mu\nu}(x)$.

Killing vectors have a very simple geometric interpretation: If the metric is *independent* of a coordinate, say, e.g., x^1 namely the transformation $x^1 \to x^1 + C$ leaves the metric unchanged, then **the associated Killing vector lies along the direction in which the metric doesn't change**. In our example, $V = \partial/\partial x^1$.

By far the most useful fact about Killing vectors is that *Killing vectors imply conserved quantities associated with the motion of free particles*. If $x^\mu(\lambda)$ is a geodesic with tangent vector $U^\mu = dx^\mu/d\lambda$, and K^μ is a Killing vector, then

$$\begin{aligned} U^\nu \nabla_\nu (K_\mu U^\mu) &= U^\nu U^\mu \nabla_\nu K_\mu + K_\mu U^\nu \nabla_\nu U^\mu \\ &= 0 \ , \end{aligned} \tag{A8}$$

where the first term vanishes from Killing's equation and the second from the fact that $x^\mu(\lambda)$ is a geodesic. Thus, the quantity $K_\mu U^\mu$ is conserved along the particle's worldline. This can be understood physically: by definition the metric is unchanging along the direction of the Killing vector. Loosely speaking, therefore, a free particle will not feel any "forces" in this direction, and the component of its momentum in that direction will consequently be conserved.

REFERENCES

[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons), chapters 7 and 13.

[2] S. Carroll, *Lecture Notes on General Relativity*, part 4. Gravitation (http://preposterousuniverse.com/grnotes/).

[3] J. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Addison-Wesley), chapter 22 and chapter 8.