The principle of equivalence and its consequences.

Asaf Pe'er[1]

March 13, 2019

This part of the course is based on Refs. [1], [2] and [3].

1. Introduction

We now turn our attention to the physics of gravitation, as described by general relativity. The mathematical formalism developed in the study of special relativity is going to be of great use - and will be extended in developing the general relativistic formalism. The discussion is naturally divided into two:

1. How the existence of matter in the universe influences space-time to create curvature (non-flat space time). The answer to this question is given by **Einstein's field equation**.

2. How do particles (including massless photons) travel in curved space-time, in such a way that we call their trajectories as being influenced by "gravity".

We will try to develop the theory from basic physical principles and argue that these lead naturally to an almost unique physical theory.

2. The Equivalence Principle

The experimental results of Eötvös and others show that the *inertial mass*, m_I is equal to the *gravitational mass*, m_g (see the introduction part). This experimental result is sometimes referred to as **the weak equivalence principle (WEP)**.

The WEP has far-reaching consequences. As noted by Einstein in his famous thought experiment, it implies that a scientist in a freely-falling (closed) elevator has **no way of measuring gravity**. Alternatively, the WEP implies that there is no way to disentangle the effects of gravitational field from those of being in a uniformly accelerating frame. (This

[1]Physics Dep., University College Cork

is in contrast to electromagnetic field, since the charge q differs from the inertial mass; in gravitational field, m_g is the gravitational "charge").

We do though have to be careful and limit our discussion to "small enough" regions in space time. If the sealed elevator was sufficiently big, the gravitational field would change from place to place inside the elevator, which could be measured. We can therefore re-state the WEP as "the laws of freely-falling particles are the same in a gravitational field and in a uniformly accelerated frame, in a small enough region of space-time".

Motivated by the equivalence of mass and energy, Einstein postulated an even stronger statement. Einstein postulated that

At every point in arbitrary gravitational field, it is possible to choose a locally inertial coordinate system, such that (within a sufficiently small region of that point) the laws of nature take the same form as in unaccelerated Cartesian coordinate system.

This is known as **the strong equivalence principle**, or the **equivalence principle**, for short. It implies that at every point in arbitrary strong gravitational field, the laws of special relativity hold locally. By "locally", we mean a region in space around the point in question in which the gravitational field can be considered (roughly) constant. It is very difficult to imagine theories which respect the WEP but not the strong equivalence principle.

The equivalence principle implies (at least, suggests) that **the action of gravity should be attributed to the curvature of space-time:** it implies that there is really no such thing as a globally "unaccelerated" (inertial) frame. One massive object in the universe is enough to provide a gravitational field, and every frame that we can imagine would be accelerated in this field. There is no such thing as "gravitationally neutral object", with respect to which we can measure the acceleration due to gravity: gravity is inescapable. Note that the equivalence of mass and energy implies that this is true for massless particles as well.

We can start by building a locally freely falling inertial frame. However, due to the inhomogeneities of gravitational field, if we try to extend this inertial frame too far, a freely falling object will look like it is "accelerating", with respect to this reference frame. Thus, from here on we will talk only about **locally inertial frames**.

2.1. Gravitational redshift

A direct consequence of the equivalence principle (with no need to get into the details of GR !) is the change in the energy of photons as they propagate in gravitational field, known as **gravitational redshift**.

Let us consider 2 observer (say "Alice" and "Bob") separated by a distance h in a uniform gravitational field with acceleration g (think of Alice as located on the top of a tower of height h above Bob).

Alice emits two photons, separated by interval $\Delta\tau_A$, as measured by her own clock (located at her position). What is the time interval $\Delta\tau_B$ between the two photons measured by Bob ?

The equivalence principle implies that $\Delta\tau_B < \Delta\tau_A$. Let us understand why. According to the equivalence principle, we can imagine Alice and Bob instead of being in a gravitational field, to be in an accelerated rocket, far in outer space where there is no gravitational field; however, the rocket is accelerated at acceleration $a = g$. **Because of the acceleration, Bob receives the signals when he is moving at a faster rate than when they were emitted.**

Let us assume that over the time of interest, we can ignore second order terms $(v/c)^2$, and $(gh/c^2)^2$ (but not first order ones). While we don't have to use this assumption, it simplifies the calculations. When neglecting $(v/c)^2$, we can neglect time dilation and Lorentz contraction, and stick with "classical" Newtonian mechanics. Our results will be good to order of gh/c^2.

As the rocket accelerates along the z axis, Bob's position is $z_B(t) = (1/2)gt^2$, while Alice's position is $z_A(t) = (1/2)gt^2 + h$. Assume that Alice sent the first photon at time $t = 0$. Bob received the first pulse at time t_1. The distance traveled by the first photon is:

$$\begin{array}{lcl} z_A(t=0) - z_B(t_1) & = & ct_1 \\ h - \frac{1}{2}gt_1^2 & = & ct_1. \end{array} \tag{1}$$

Alice emits the second photon at time $t = \Delta\tau_A$, while Bob receives it at time $\Delta\tau_B$ after receiving the first photon, which is $t = t_1 + \Delta\tau_B$. The distance traveled by the second photon is thus

$$\begin{array}{lcl} z_A(\Delta\tau_A) - z_B(t_1 + \Delta\tau_B) & = & c(t_1 + \Delta\tau_B - \Delta\tau_A) \\ h + \frac{1}{2}g\Delta\tau_A^2 - \frac{1}{2}g\left(t_1 + \Delta\tau_B\right)^2 & = & c(t_1 + \Delta\tau_B - \Delta\tau_A) \\ h - \frac{1}{2}gt_1^2 - gt_1\Delta\tau_B + \frac{1}{2}g(\Delta\tau_A^2 - \Delta\tau_B^2) & = & c(t_1 + \Delta\tau_B - \Delta\tau_A) \\ h - \frac{1}{2}gt_1^2 - gt_1\Delta\tau_B & \simeq & c(t_1 + \Delta\tau_B - \Delta\tau_A) \end{array} \tag{2}$$

where in the last line we assumed that $\Delta\tau_A$ is small, so we kept only linear terms in $\Delta\tau_A$, $\Delta\tau_B$. Subtracting Equation 1 from Equation 2, we are left with

$$-gt_1\Delta\tau_B \simeq c(\Delta\tau_B - \Delta\tau_A) \tag{3}$$

Using $t_1 \simeq h/c$ (allowed, as we neglect terms of second order and higher), we eventually get

$$\Delta\tau_B \simeq \Delta\tau_A \left(\frac{1}{1+\frac{gh}{c^2}}\right) \simeq \Delta\tau_A \left(1 - \frac{gh}{c^2}\right) \tag{4}$$

Thus, the interval in which two photons are received is smaller by a factor $\approx (1 - gh/c^2)$ than the interval in which they were emitted. The equivalence principle tells us that exactly the same effect occurs in a uniform gravitational field. Since gh is the difference in gravitational potential, $\Delta\Phi$, we can write $\Delta\tau_B \simeq \Delta\tau_A(1 - \Delta\Phi/c^2)$. When the receiver is in *lower* gravitational potential (=deeper in the gravitational field) than the emitter, the signal will be received more quickly than emitted. If, on the other hand, the receiver is in *higher* potential, the signal will be received more slowly than emitted. This had been tested experimentally.

Similarly, the crest of a light wave of frequency ν can be thought of as a series of signals emitted at the rate which is equal to the frequency of the wave. Thus, Equation 4 can be applied for light. Since $\nu \propto \Delta\tau^{-1}$, if a light is emitted at frequency $\nu_\star$ from the surface of a star, its observed frequency by a distant observer will be

$$\nu_\infty = \nu_\star\left(1 - \frac{\Delta\Phi}{c^2}\right) = \nu_\star\left(1 - \frac{GM}{Rc^2}\right), \tag{5}$$

where M is the mass of the star and R is its radius. (This is of course accurate only to first order, namely to small values of GM/Rc^2). This frequency is **less** than its emitted frequency; alternatively, the wavelength of the light $\lambda = c/\nu$ **increases** as it leaves the gravitational potential, which shifts it towards the *red* part of the spectrum. Hence the term "red-shift".

Note: don't confuse the gravitational redshift caused by photon propagation from deeper gravitational potential to a shallower one with the *cosmological redshift* caused by the expansion of the universe. This will be discussed at the later stages of the course.

A few examples are given in table 1.

3. Mathematical description of curved space time

In the example given above of a gravitational redshift, the two photons follow exactly the same paths in space-time. Thus, simple geometry tells us that the time intervals, $\Delta\tau_A$

object	R [cm]	M [gr]	GM/Rc^2
Earth	6×10^8	6×10^{27}	10^{-9}
Sun	10^{11}	10^{33}	10^{-6}
White dwarf	10^9	10^{33}	10^{-4}
Neutron star	10^6	10^{33}	10^{-1}

Table 1: Typical values of gravitational redshift.

and $\Delta\tau_B$ should be the same; however, we saw that this is not the case. The reason is of course that space-time cannot be described by "simple geometry". A better description is that **in the presence of gravity, space-time is curved**. We shell proceed by adopting this assumption of curved space-time, and see where it leads us.

The principle of equivalence tells us that the laws of physics, in small enough regions of space-time look like those of special relativity. We can thus start by looking at an arbitrary point in space time and set a freely-falling coordinate system ξ^α, in which the laws of special relativity hold. Thus, we can write the proper time as

$$d\tau^2 = -\frac{1}{c^2}\eta_{\alpha\beta}d\xi^\alpha d\xi^\beta \tag{6}$$

This coordinate system is (locally) inertial; thus, if no forces act on a particle, its trajectory is given by

$$\frac{dU^\alpha}{d\tau} = \frac{d^2\xi^\alpha}{d\tau^2} = 0, \tag{7}$$

where $U^\alpha \equiv d\xi^\alpha/d\tau$ is the four velocity (see chapter on SR, equations (62) and (57)).

We now want to express the proper time in any *arbitrary* coordinate system, x^μ. Using the chain rule, we write

$$d\tau^2 = -\frac{1}{c^2}\eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}dx^\mu\frac{\partial\xi^\beta}{\partial x^\nu}dx^\nu = -\frac{1}{c^2}g_{\mu\nu}dx^\mu dx^\nu \tag{8}$$

where $g_{\mu\nu}$ is the **metric tensor**, defined as

$$g_{\mu\nu} \equiv \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu}. \tag{9}$$

In our new system, x^μ, being arbitrary, a free-falling particles does seem to be accelerated. Let us see how the acceleration looks. We write Equation 7 in our new coordinate system:

$$\begin{aligned} 0 = \frac{dU^\alpha}{d\tau} &= \frac{d}{d\tau}\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\right) \\ &= \left(\left(\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\right)\left(\frac{dx^\nu}{d\tau}\right)\right)\left(\frac{dx^\mu}{d\tau}\right) + \left(\frac{\partial\xi^\alpha}{\partial x^\mu}\right)\left(\frac{d^2x^\mu}{d\tau^2}\right) \end{aligned} \tag{10}$$

We now multiply Equation 10 by $\partial x^\lambda/\partial\xi^\alpha$ and use the product rule,

$$\left(\frac{\partial\xi^\alpha}{\partial x^\mu}\right)\left(\frac{\partial x^\lambda}{\partial\xi^\alpha}\right) = \delta^\lambda_\mu \tag{11}$$

to write the equation of motion as

$$0 = \frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{12}$$

Here, $\Gamma^\lambda_{\mu\nu}$ is the **affine connection**, defined by

$$\Gamma^\lambda_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}. \tag{13}$$

We can write the equation of motion in a form that clarifies the acceleration term,

$$\frac{du^\lambda}{d\tau} = -\Gamma^\lambda_{\mu\nu}u^\mu u^\nu, \tag{14}$$

where $u^\mu \equiv dx^\mu/d\tau$ is the 4-velocity in the new (non-inertial) frame.

Note. Mathematicaly, the affine connection gives us a way of relating vectors in the tangent spaces of nearby points, hence its name. We havn't yet defined that properly, and so we will return to this point later.

If we are dealing with the motion of massless particles, the equation of motion takes the same form as Equations 7, 8 and 10, only that τ needs to be replaced with some arbitrary parameter σ, as $d\tau$ is always zero along the world line (see discussion in §4.3 in the chapter on SR). Thus, we have

$$\begin{array}{l} \frac{dU^\alpha}{d\sigma} = \frac{d^2\xi^\alpha}{d\sigma^2} = 0, \\ 0 = -g_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}, \\ 0 = \frac{d^2x^\lambda}{d\sigma^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} \end{array} \tag{15}$$

(compare to Equations 7, 8, 12). Equations 14, 15 are known as the **geodesic equation**. These describe the motion of a free particle, as measured by arbitrary coordinate system.

4. The relation between the affine connection and the metric tensor

Let us see what we have done so far. We started from an inertial frame (ξ^μ), in which a particle is freely-falling, as no acceleration acts on it. We then changed to a different coordinate system (x^μ), in which the particle seems to accelerate (Equation 14). Up to this point, we just did a **mathematical** operation of coordinate changes.

However, we would also like to give a **physical** interpretation: the acceleration, described by the affine connection $\Gamma^{\lambda}_{\mu\nu}$, will be interpreted as due to the gravitational force. Thus, the affine connection represents, in some way, the gravitational force (although it has a broader mathematical definition).

In this new frame, the proper time interval $d\tau$ is determined by the metric tensor, $g_{\mu\nu}$ (see Equation 8). Thus, there should be an obvious connection between the metric tensor $g_{\mu\nu}$ and the affine connection $\Gamma^{\lambda}_{\mu\nu}$. In fact, as we show here, the affine connection is determined by the derivatives of the metric tensor. Hence, the metric tensor also serves as the *gravitational potential* (whose derivative determines the force).

The calculation is done as follows. We begin by the definition of the metric tensor in Equation 9. We differentiate with respect to x^{λ}, to get

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \frac{\partial^2 \xi^{\alpha}}{\partial x^{\lambda}\partial x^{\mu}}\frac{\partial \xi^{\beta}}{\partial x^{\nu}}\eta_{\alpha\beta} + \frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\frac{\partial^2 \xi^{\beta}}{\partial x^{\lambda}\partial x^{\nu}}\eta_{\alpha\beta} \tag{16}$$

Using the definition of the affine connection in Equation 13, we can write

$$\frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu}\partial x^{\nu}} = \Gamma^{\lambda}_{\mu\nu}\frac{\partial \xi^{\alpha}}{\partial x^{\lambda}} \tag{17}$$

[formally, Equation 17 is obtained from Equation 13 by using the product rule, $(\partial\xi^{\beta}/\partial x^{\lambda})(\partial x^{\lambda}/\partial\xi^{\alpha}) = \delta^{\beta}_{\alpha}$]. Using Equation 17 in Equation 16 gives

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \Gamma^{\rho}_{\lambda\mu}\frac{\partial \xi^{\alpha}}{\partial x^{\rho}}\frac{\partial \xi^{\beta}}{\partial x^{\nu}}\eta_{\alpha\beta} + \Gamma^{\rho}_{\lambda\nu}\frac{\partial \xi^{\alpha}}{\partial x^{\mu}}\frac{\partial \xi^{\beta}}{\partial x^{\rho}}\eta_{\alpha\beta} = \Gamma^{\rho}_{\lambda\mu}g_{\rho\nu} + \Gamma^{\rho}_{\lambda\nu}g_{\rho\mu}, \tag{18}$$

where we used again the definition of $g_{\mu\nu}$ in Equation 9.

We now add to Equation 18 the same Equation with μ and λ interchanged, and subtract the same equation with ν and λ interchanged. We get:

$$\begin{aligned} \tfrac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \tfrac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \tfrac{\partial g_{\mu\lambda}}{\partial x^{\nu}} &= \Gamma^{\kappa}_{\lambda\mu}g_{\kappa\nu} + \Gamma^{\kappa}_{\lambda\nu}g_{\kappa\mu} \\ &\quad +\Gamma^{\kappa}_{\lambda\mu}g_{\kappa\nu} + \Gamma^{\kappa}_{\mu\nu}g_{\kappa\lambda} \\ &\quad -\Gamma^{\kappa}_{\nu\mu}g_{\kappa\lambda} - \Gamma^{\kappa}_{\lambda\nu}g_{\kappa\mu} \\ &= 2\Gamma^{\kappa}_{\lambda\mu}g_{\kappa\nu}, \end{aligned} \tag{19}$$

where we have used the fact that both $\Gamma^{\kappa}_{\nu\mu}$ and $g_{\mu\nu}$ are symmetric under interchange of μ and ν. We now define the matrix $g^{\mu\nu}$ as the inverse of the matrix $g_{\mu\nu}$, namely

$$g^{\nu\sigma}g_{\nu\kappa} = \delta^{\sigma}_{\kappa}, \tag{20}$$

and multiply Equation 20 by $(1/2)g^{\nu\sigma}$, to get

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2}g^{\nu\sigma}\left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}}\right). \tag{21}$$

The right hand side of Equation 21 is often called a **Christoffel symbol**, and denoted by

$$\left\{ \begin{matrix} \sigma \\ \lambda\mu \end{matrix} \right\} \tag{22}$$

Example. Using Equation 8, the metric can be defined on any space. As a siple example, consider a 2-dimensional flat space, written in r, θ coordinates. An element of length is $dS^2 = dr^2 + r^2 d\theta^2$, and thus the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{23}$$

Hence,

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}, \tag{24}$$

and the only non-zero components of the affine connection are

$$\Gamma^r_{\theta\theta} = -r \;;\; \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \tag{25}$$

We will continue with this example below.

5. The geodesic equation and the variational principle

Let us consider the motion of a very small (test) particle moving in a gravitational field. By *test particle*, we mean a particle whose mass is so small that it produces no spacetime curvature by itself.

We consider no forces acting on the particle apart from gravitational force (namely, no EM forces, etc.). Such a particle is called *"freely falling"* or *"free"* for short. Note the difference between GR and Newtonian mechanics: In Newtonian mechanics, a "free" particle is not influenced by *any* force, including gravitational force, while in GR gravity is included; this is because gravity is not considered as a "force" but as a curvature in space time.

The general principle for motion of a free (massive) test particle is not changed between SR and GR: a particle takes **the shortest path**, in the sense that the path the particles moves along between points A and B is the path that **extremize (minimize) the proper time between the points**. This is a generalization of the concept of a "straight line" in Euclidean space: a "straight line" is the path of shortest distance between two points. Such paths are known as **geodesics**, and the equation of motion of a free particle is called **the geodesic equation**.

The treatment is very similar to the treatment done in SR; the difference would be that the proper time is given by Equation 8 rather than Equation 6. Let us work first the SR treatment, although the answer should be clear already.

5.1. Equation of motion of free particle in SR from the variational principle

We want to find the path that extremizes τ. We thus write

$$\tau_{A-B} = \int_A^B d\tau = \int d\lambda \left[\left(-\frac{1}{c^2}\right)\eta_{\alpha\beta}\frac{d\xi^\alpha}{d\lambda}\frac{d\xi^\beta}{d\lambda}\right]^{1/2}, \tag{26}$$

where λ is a parameter along the world line of the particle. We seek the world line that extremizes τ_{A-B}. For that, we can use the variation principle: finding the path for which τ_{A-B} does not change when a small change $\delta\xi^\alpha(\lambda)$ occurs. This is an identical problem to problems studied in Newtonian mechanics, where the integrand plays the role of the Lagrangian, ξ^α is the dynamical variable and $d\lambda$ is the time.

We can thus write Lagrange's equation of motion,

$$-\frac{d}{d\lambda}\left(\frac{\partial L}{\partial\left(\frac{d\xi^\sigma}{d\lambda}\right)}\right) + \frac{\partial L}{\partial \xi^\sigma} = 0, \tag{27}$$

where

$$L \equiv \left(-\frac{1}{c^2}\eta_{\alpha\beta}\frac{d\xi^\alpha}{d\lambda}\frac{d\xi^\beta}{d\lambda}\right)^{1/2}, \tag{28}$$

and thus $\partial L/\partial\xi^\sigma = 0$.

The Equation of motion (Equation 27) thus becomes

$$\frac{d}{d\lambda}\left(\frac{1}{2L}\left(-\frac{1}{c^2}\right)2\eta_{\sigma\beta}\frac{d\xi^\beta}{d\lambda}\right) = 0 \tag{29}$$

where the extra factor of 2 comes from the symmetry between α and β, and use was made of δ^α_σ. Since $L = d\tau/d\lambda$, we obtain

$$\frac{d}{d\lambda}\left(\frac{d\xi^\beta}{d\tau}\right) = 0, \tag{30}$$

and multiplying by $d\lambda/d\tau$ we get the familiar result,

$$\frac{d^2\xi^\beta}{d\tau^2} = 0. \tag{31}$$

Not surprisingly, this is identical to Equation (62) in the chapter on SR (without forces).

5.2. Equation of motion of free particle in GR from the variational principle

In a completely analogue way to the special relativistic case, the motion of a free particle in curved space time is given by taking the extremum of

$$\tau_{A-B} = \int_A^B d\tau = \int d\lambda \left[\left(-\frac{1}{c^2} \right) g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right]^{1/2}, \tag{32}$$

The Lagrangian is

$$L \equiv \left(-\frac{1}{c^2} g_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \right)^{1/2}, \tag{33}$$

and Lagrange's equation becomes

$$\begin{aligned} \tfrac{d}{d\lambda}\left(\tfrac{1}{2L} \left(-\tfrac{1}{c^2} \right) 2 g_{\sigma\beta} \tfrac{dx^\beta}{d\lambda} \right) &= -\tfrac{1}{c^2} \tfrac{1}{2L} \left(\tfrac{\partial g_{\alpha\beta}}{\partial x^\sigma} \tfrac{dx^\alpha}{d\lambda} \tfrac{dx^\beta}{d\lambda} \right) \\ \tfrac{d}{d\tau}\left(g_{\sigma\beta} \tfrac{dx^\beta}{d\tau} \right) &= \tfrac{1}{2} \tfrac{\partial g_{\alpha\beta}}{\partial x^\sigma} \tfrac{dx^\alpha}{d\tau} \tfrac{dx^\beta}{d\tau} \end{aligned} \tag{34}$$

where in the last line we multiplied both sides by $d\lambda/d\tau$. We can thus write the equation of motion as

$$\frac{\partial g_{\sigma\beta}}{\partial x^\mu} \frac{dx^\mu}{d\tau} \frac{dx^\beta}{d\tau} + g_{\sigma\beta} \frac{d^2 x^\beta}{d\tau^2} - \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \tag{35}$$

or by noticing the symmetry in α and β and re-shuffling dummy indices in the first term,

$$g_{\sigma\beta} \frac{d^2 x^\beta}{d\tau^2} + \frac{1}{2} \left(\frac{\partial g_{\sigma\beta}}{\partial x^\alpha} + \frac{\partial g_{\alpha\sigma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\sigma} \right) \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \tag{36}$$

multiplying by the inverse of the metric $g^{\sigma\rho}$, we obtain the equation of motion for a free particle, or the **geodesic equation in curved space time**,

$$\frac{d^2 x^\rho}{d\tau^2} + \Gamma^\rho_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \tag{37}$$

(not surprising, this is identical to Equation 14). Alternatively, Equation 37 could be written in the form

$$\frac{du^\rho}{d\tau} + \Gamma^\rho_{\alpha\beta} u^\alpha u^\beta = 0. \tag{38}$$

Similarly, photons travel along **null worldline**, namely $d\tau = 0$. Their path can be parameterized by a parameter λ, so that $x^\alpha = x^\alpha(\lambda)$ along their world line. Their equation of motion, which is known as **the geodesic equation for null geodesics** is

$$\frac{d^2 x^\rho}{d\lambda^2} = -\Gamma^\rho_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} \tag{39}$$

Null curves that satisfy Equation 39 are known as **null geodesics**. Light rays move on null geodesics. Note that $\lambda \neq \tau$, since $\tau = 0$ along null geodesics !. Rather, λ is a parameter describing the path of a photon.

6. The Newtonian limit (weak field approximation)

Let us now see what form the geodesic equation obtains in the weak field (=Newtonian limit) approximation. Obviously, we expect to retrieve the familiar Newtonian result, otherwise, we are in trouble !.

We define "Newtonian limit" by three requirements: (i) the particles are moving slowly (with respect to the speed of light), (ii) the gravitational field is weak (can be considered a perturbation of flat space), and (iii) the field is also static (unchanging with time). Let us see what these assumptions do to the geodesic equation (of a massive particle). "Moving slowly" means that

$$\frac{dx^i}{d\tau} << \frac{cdt}{d\tau} \ , \tag{40}$$

so the geodesic equation (Equation 37) becomes

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{cdt}{d\tau}\right)^2 = 0 \ . \tag{41}$$

Since the field is static, all time derivatives of $g_{\mu\nu}$ vanish, and the relevant Christoffel symbols Γ^μ_{00} simplify:

$$\begin{aligned} \Gamma^\mu_{00} &= \tfrac{1}{2}g^{\mu\lambda}(\partial_0 g_{\lambda 0} + \partial_0 g_{0\lambda} - \partial_\lambda g_{00}) \\ &= -\tfrac{1}{2}g^{\mu\lambda}\partial_\lambda g_{00} \ . \end{aligned} \tag{42}$$

(Note that we used the notation $\partial_\mu g_{\lambda\sigma} \equiv \partial g_{\lambda\sigma}/\partial x^\mu$, see SR, Equation 46). Finally, the weakness of the gravitational field allows us to decompose the metric into the Minkowski form plus a small perturbation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \qquad |h_{\mu\nu}| << 1 \ . \tag{43}$$

(We are working in Cartesian coordinates, so $\eta_{\mu\nu}$ is the canonical form of the metric. The "smallness condition" on the metric perturbation $h_{\mu\nu}$ doesn't really make sense in other coordinates.) From the definition of the inverse metric, $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma$, we find that to first order in h,

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \ , \tag{44}$$

where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. In fact, we can use the Minkowski metric to raise and lower indices on an object of any definite order in h, since the corrections would only contribute at higher orders.

Putting it all together, we find

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00} \ . \tag{45}$$

The geodesic equation (37) is therefore

$$\frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\lambda}\partial_\lambda h_{00}\left(\frac{cdt}{d\tau}\right)^2 \ . \tag{46}$$

Using $\partial_0 h_{00} = 0$ (stationary gravitational field), the $\mu = 0$ component of this is just

$$\frac{d^2t}{d\tau^2} = 0 \ . \tag{47}$$

That is, $\frac{dt}{d\tau}$ is constant. To examine the spacelike components of Equation 46, recall that the spacelike components of $\eta^{\mu\nu}$ are just those of a 3×3 identity matrix. We therefore have

$$\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\left(\frac{cdt}{d\tau}\right)^2 \partial_i h_{00} \ . \tag{48}$$

Dividing both sides by $\left(\frac{cdt}{d\tau}\right)^2$ has the effect of converting the derivative on the left-hand side from τ to t, leaving us with

$$\frac{d^2x^i}{c^2dt^2} = \frac{1}{2}\partial_i h_{00} \ . \tag{49}$$

This begins to look like recovering Newton's theory of gravitation. In fact, we can identify

$$h_{00} = -\frac{2\Phi}{c^2} \ , \tag{50}$$

to get the familiar result

$$\frac{d^2x^i}{dt^2} = -\partial_i\Phi. \tag{51}$$

In other words, we got

$$g_{00} = -\left(1 + \frac{2\Phi}{c^2}\right) \ . \tag{52}$$

Therefore, we have shown that the curvature of spacetime is indeed sufficient to describe gravity in the Newtonian limit, as long as the metric takes the form of Equation 50.

We still need to find field equations for the metric which imply that this is the form taken, and that for a single gravitating body we recover the Newtonian formula

$$\Phi = -\frac{GM}{r} \ . \tag{53}$$

This will be done shortly.

REFERENCES

[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons), chapter 4.

[2] S. Carroll, *Lecture Notes on General Relativity*, part 4. Gravitation (http://preposterousuniverse.com/grnotes/).

[3] J. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Addison-Wesley), chapter 6.