Free Fields

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After discussing classical fields, we turn our attention to quantum fields; we shift from $\hbar = 0$ to $\hbar = 1$...

1. Canonical Quantization

In quantum mechanics, **canonical quantization** is a recipe that takes us from the Hamiltonian formalism of classical dynamics to the quantum theory. The recipe tells us to **take the generalized coordinates** q_a **and their conjugate momenta** p^a **and promote them to operators**. The Poisson bracket structure of classical mechanics morphs into the structure of commutation relations between **operators**, so that

$$\begin{array}{c} [q_a, q_b] = \left[p^a, p^b\right] = 0 \\ \left[q_a, p^b\right] = i\delta_a^b, \end{array} \tag{1}$$

where q_a and p^a are now operators (and $\hbar = 1$).

In field theory we do the same, now for the field $\phi_a(\vec{x})$ **and its momentum conjugate** $\pi^b(\vec{x})$. Thus a **quantum field** is an operator-valued function of space (not space-time!), obeying the commutation relations

$$\begin{array}{c} [\phi_a(\vec{x}), \phi_b(\vec{y})] = \left[\pi^a(\vec{x}), \pi^b(\vec{y})\right] = 0 \\ \left[\phi_a(\vec{x}), \pi^b(\vec{y})\right] = i\delta^{(3)}(\vec{x} - \vec{y})\delta_a^b. \end{array} \tag{2}$$

Pay attention that as opposed to QM, in QFT we treat $\vec{x}$ and $\vec{y}$ as **labels** - similarly to a and b (that is, we do not promote them to operators!).

Note that we have lost all track of Lorentz invariance since we have separated space $\vec{x}$ and time t; This was neccesary once we introduced momenta.

We are working in the Schrödinger picture so that the operators $\phi_a(\vec{x})$ and $\pi^a(\vec{x})$ depend on space but do not depend on time at all. **All time dependence sits in the states** $|\psi\rangle$**, which evolve by the usual Schrödinger equation**

$$i\frac{d|\psi\rangle}{dt} = H|\psi\rangle. \tag{3}$$

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We are not doing anything different from usual quantum mechanics; we are merely applying the old formalism to fields. Be warned however that the notation $|\psi\rangle$ for the state is deceptively simple: if you were to write the wavefunction in quantum field theory, it would be a *functional*, namely, a function that associate a (complex) number to every possible configuration of the field ϕ.

The typical information we want to know about a quantum theory is **the spectrum of the Hamiltonian** H. In quantum field theories, this is usually **very hard**. One reason for this is that we have an infinite number of degrees of freedom - at least one for every point $\vec{x}$ in space.

However, for certain theories - known as **free field theories** - we can find a way to write the dynamics such that each degree of freedom evolves independently from all the others. *Free field theories typically have Lagrangians which are quadratic in the fields, so that the equations of motion are linear.*

For example, the simplest relativistic free theory is the **classical Klein-Gordon (KG) equation** for a real scalar field $\phi(\vec{x}, t)$,

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \tag{4}$$

To exhibit the coordinates in which the degrees of freedom decouple from each other, we need only take the Fourier transform:

$$\phi(\vec{x}, t) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi(\vec{p}, t). \tag{5}$$

Then $\phi(\vec{p}, t)$ satisfies:

$$\left(\frac{\partial^2}{\partial t^2} + (\vec{p}^2 + m^2) \right) \phi(\vec{p}, t) = 0. \tag{6}$$

We thus find that for each value of $\vec{p}$, $\phi(\vec{p}, t)$ evolves independently, and solves the equation of a harmonic oscillator, vibrating at frequency

$$\boxed{\omega_{\vec{p}} = +\sqrt{\vec{p}^2 + m^2}.} \tag{7}$$

We thus find that **the most general solution to the KG equation is a linear superposition of simple harmonic oscillators**, each vibrating at a different frequency with a different amplitude. To quantize $\phi(\vec{x}, t)$ we must simply quantize this infinite number of harmonic oscillators. Let's recall how to do this.

1.1. The Simple Harmonic Oscillator

Consider the quantum mechanical Hamiltonian:

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2, \tag{8}$$

with q and p being operators obeying the canonical commutation relation $[q, p] = i$. To find the spectrum we define the creation and annihilation operators (also known as raising/lowering operators, or sometimes ladder operators),

$$a = \sqrt{\frac{\omega}{2}}q + \frac{i}{\sqrt{2\omega}}p; \qquad a^\dagger = \sqrt{\frac{\omega}{2}}q - \frac{i}{\sqrt{2\omega}}p. \tag{9}$$

This can easily be inverted to give

$$q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger); \qquad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \tag{10}$$

Substituting, we find

$$\left[a, a^\dagger\right] = -\frac{i}{2}\left[q, p\right] + \frac{i}{2}\left[p, q\right] = 1 \tag{11}$$

and the Hamiltonian can be written as

$$H = \frac{1}{2}\omega(aa^\dagger + a^\dagger a) = \omega\left(a^\dagger a + \frac{1}{2}\right). \tag{12}$$

The commutators between the Hamiltonian and the creation and annihilation operators are given by

$$\left[H, a^\dagger\right] = \omega a^\dagger; \qquad [H, a] = -\omega a. \tag{13}$$

These relations imply that a and $a^\dagger$ take us between energy eigenstates. Let $|E\rangle$ be an eigenstate with energy E, so that $H|E\rangle = E|E\rangle$. Then we can construct more eigenstates by acting with a and $a^\dagger$,

$$Ha^\dagger|E\rangle = (E + \omega)a^\dagger|E\rangle; \qquad Ha|E\rangle = (E - \omega)a|E\rangle. \tag{14}$$

The system thus has ladder states, with energies

$$..., E - \omega, E, E + \omega, E + 2\omega, ... \tag{15}$$

If the energy is bounded from below, there must be a *ground state* $|0\rangle$, which satisfies $a|0\rangle = 0$. This state has ground state energy (also known as zero point energy),

$$H|0\rangle = \frac{1}{2}\omega|0\rangle. \tag{16}$$

Excited states then arise from repeated application of $a^\dagger$,

$$|n\rangle = (a^\dagger)^n|0\rangle \quad \text{with} \quad H|n\rangle = \left(n + \frac{1}{2}\right)\omega|n\rangle \tag{17}$$

(I have ignored the normalization of these states, and thus in Equation 17, $\langle n|n\rangle \neq 1$).

2. The Free Scalar Field

We now apply the quantization of the harmonic oscillator to the free scalar field (the Klein-Gordon field). We write ϕ and π as a linear sum of an infinite number of creation and annihilation operators $a^\dagger_{\vec{p}}$ and $a_{\vec{p}}$, indexed by the 3-momentum $\vec{p}$,

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left[a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right], \tag{18}$$

and

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left[a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right]. \tag{19}$$

(Equations 18 and 19 can be viewed as the **definitions of the operators** $a_{\vec{p}}$ **and** $a^\dagger_{\vec{p}}$). Further note that $\phi(x)$ and $\pi(x)$ are Hermitian operators being real scalar fields, and thus $a^\dagger_{\vec{p}}$ is indeed the Hermitian conjugate of $a_{\vec{p}}$.

Claim: The commutation relations for ϕ and π are equivalent to the following commutation relations for $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$

$$\begin{array}{c} [\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \end{array} \quad \Leftrightarrow \quad \begin{array}{c} [a_{\vec{p}}, a_{\vec{q}}] = \left[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}} \right] = 0 \\ \left[a_{\vec{p}}, a^\dagger_{\vec{q}} \right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}). \end{array} \tag{20}$$

Proof: We will work it just one way. Assume that $\left[a_{\vec{p}}, a^\dagger_{\vec{q}} \right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})$. Then

$$\begin{aligned} [\phi(\vec{x}), \pi(\vec{y})] &= \int \tfrac{d^3p d^3q}{(2\pi)^6} \tfrac{(-i)}{2} \sqrt{\tfrac{\omega_{\vec{q}}}{\omega_{\vec{p}}}} \left(- \left[a_{\vec{p}}, a^\dagger_{\vec{q}} \right] e^{i\vec{p}\cdot\vec{x} - i\vec{q}\cdot\vec{y}} + \left[a^\dagger_{\vec{p}}, a_{\vec{q}} \right] e^{-i\vec{p}\cdot\vec{x} + i\vec{q}\cdot\vec{y}} \right) \\ &= \int \tfrac{d^3p}{(2\pi)^3} \tfrac{(-i)}{2} \left(-e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{i\vec{p}\cdot(\vec{y}-\vec{x})} \right) \\ &= i\delta^{(3)}(\vec{x} - \vec{y}) \end{aligned} \tag{21}$$

which proves the claim.

The Hamiltonian.

Let us now compute the Hamiltonian in terms of $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$. We have

$$\begin{aligned} H &= \tfrac{1}{2} \int d^3x \left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2 \right] \\ &= \tfrac{1}{2} \int \tfrac{d^3x d^3p d^3q}{(2\pi)^6} \Big[-\tfrac{\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}}{2} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right) \left(a_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} - a^\dagger_{\vec{q}} e^{-i\vec{q}\cdot\vec{x}} \right) \\ &\quad + \tfrac{1}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}} \left(i\vec{p} a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - i\vec{p} a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right) \cdot \left(i\vec{q} a_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} - i\vec{q} a^\dagger_{\vec{q}} e^{-i\vec{q}\cdot\vec{x}} \right) \\ &\quad + \tfrac{m^2}{2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right) \left(a_{\vec{q}} e^{i\vec{q}\cdot\vec{x}} + a^\dagger_{\vec{q}} e^{-i\vec{q}\cdot\vec{x}} \right) \Big] \\ &= \tfrac{1}{4} \int \tfrac{d^3p}{(2\pi)^3} \tfrac{1}{\omega_{\vec{p}}} \left[\left(-\omega^2_{\vec{p}} + \vec{p}^2 + m^2 \right) \left(a_{\vec{p}} a_{-\vec{p}} + a^\dagger_{\vec{p}} a^\dagger_{-\vec{p}} \right) + \left(\omega^2_{\vec{p}} + \vec{p}^2 + m^2 \right) \left(a_{\vec{p}} a^\dagger_{\vec{p}} + a^\dagger_{\vec{p}} a_{\vec{p}} \right) \right]. \end{aligned}$$

To get from the first to the second line, I have inserted the expressions for ϕ and π given in Equations 18 and 19. The last line is obtained by first integrating over d^3x to obtain delta-functions $\delta^{(3)}(\pm\vec{p}\pm\vec{q})$ (see "preliminaries", Equation 30). Using these delta-functions, integration over d^3q is performed. Finally, the last line is obtained by using the fact that $\omega_{-\vec{p}} = \omega_{\vec{p}}$ (see Equation 7).

We can now use the expression for the frequency given in Equation 7, $\omega_{\vec{p}}^2 = \vec{p}^2 + m^2$, to eliminate the first term in the last line. We are thus left with

$$\begin{aligned} H &= \tfrac{1}{2}\int \tfrac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\left[(a_{\vec{p}}a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger a_{\vec{p}}\right] \\ &= \int \tfrac{d^3p}{(2\pi)^3}\omega_{\vec{p}}\left[a_{\vec{p}}^\dagger a_{\vec{p}} + \tfrac{1}{2}(2\pi)^3\delta^{(3)}(0)\right], \end{aligned} \tag{22}$$

where in the last line we used the commutation relation in Equation 20 (and $\vec{p} - \vec{p} = 0$).

OK. The second term in the Hamiltonian is proportional to a delta-function, *evaluated at zero where it has its infinite spike*. Moreover, *the integral over $\omega_{\vec{p}}$ diverges at large p*. Welcome to the world of QFT...

What to do? Let's start by looking at the ground state where this infinity first becomes apparent.

3. The Vacuum

Following our procedure for the harmonic oscillator, let's define the **vacuum state** $|0\rangle$ by insisting that it is annihilated by *all* $a_{\vec{p}}$,

$$a_{\vec{p}}|0\rangle = 0 \qquad \forall\vec{p} \tag{23}$$

Using this definition, the energy E_0 of the ground state comes from the second term in Equation 22,

$$H|0\rangle \equiv E_0|0\rangle = \left[\int d^3p\frac{1}{2}\omega_{\vec{p}}\delta^{(3)}(0)\right]|0\rangle = \infty|0\rangle \tag{24}$$

The subject of quantum field theory is full with infinities. Each tells us something important, usually that we are doing something wrong, or asking the wrong question. Let's take some time to explore where this infinity comes from and how we should deal with it.

A careful look reveals that in fact **there are two different ∞'s** lurking in Equation 24. The first arises because space is infinitely large. (Infinities of this type are often referred to as **infra-red divergences** although in this case the ∞ is so simple that it barely deserves this name).

To extract out this infinity, let's consider putting the theory in a box with sides of length L. We impose periodic boundary conditions on the field. Then, taking the limit where $L \to \infty$, we get

$$(2\pi)^3\delta^{(3)}(0) = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3x e^{i\vec{x}\cdot\vec{p}}\bigg|_{\vec{p}=0} = \lim_{L\to\infty}\int_{-L/2}^{L/2} d^3x = V, \tag{25}$$

where V is the volume of the box. We thus find that the $\delta(0)$ divergence arises because we are computing the **total energy**, rather than the **energy density**, $\mathcal{E}_0$. To find $\mathcal{E}_0$, we can simply divide by the volume,

$$\mathcal{E}_0 \equiv \frac{E_0}{V} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2}\omega_{\vec{p}}. \tag{26}$$

This integral, however, *is still infinite*. We recognize it as the sum of ground state energies for each harmonic oscillator. But $\mathcal{E}_0 \to \infty$, due to the $|\vec{p}| \to \infty$ limit of the integral. This is a high frequency - or short distance - infinity known as an **ultra-violet divergence**.

This divergence arises because of our hubris. We have assumed that our theory is valid to arbitrarily short distance scales, corresponding to arbitrarily high energies. This is clearly absurd. The integral should be cut-off at high momentum in order to reflect the fact that our theory is likely to break down in some way.

Practically, we can deal with the infinities in Equation 24 as follows. In physics we are only interested in **energy differences**. There is no way to measure E_0 directly, so we can simply redefine the Hamiltonian by subtracting off this infinity,

$$H = \int \frac{d^3p}{(2\pi)^3}\omega_{\vec{p}}{a^\dagger}_{\vec{p}}a_{\vec{p}}, \tag{27}$$

so that, with this new definition, $H|0\rangle = 0$.

In fact, the difference between this Hamiltonian and the previous one is merely an ordering ambiguity in moving from the classical theory to the quantum theory. For example, if we defined the Hamiltonian of the harmonic oscillator to be $H = (1/2)(\omega q - ip)(\omega q + ip)$, which is classically the same as our original choice (compare to Equation 8), then upon quantization it would naturally give $H = \omega a^\dagger a$, as in Equation 27.

This type of ordering ambiguity arises a lot in field theories. We will come across a number of ways of dealing with it. The method that we have used above is called **normal ordering**.

Definition. We write the **normal ordered** string of operators $\phi_1(\vec{x}_1)...\phi_n(\vec{x}_n)$ as

$$:\phi_1(\vec{x}_1)...\phi_n(\vec{x}_n): \tag{28}$$

It is defined to be the usual product with **all annihilation operators $a_{\vec{p}}$ placed to the right**. So, for the Hamiltonian, we could write Equation 27 as

$$: H := \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} a^{\dagger}{}_{\vec{p}} a_{\vec{p}}, \tag{29}$$

In the remainder of this section, we will normal order all operators in this manner.

3.1. The Cosmological Constant.

We claimed above that "there is no way to measure E_0 directly". This is, how to say, not exactly true: gravity is supposed to see everything! The sum of all the zero point energies should contribute to the stress-energy tensor that appears on the right-hand side of Einstein's equations. We expect them to appear as a **cosmological constant**, $\Lambda = E_0/V$,

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = -8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}. \tag{30}$$

Current observations suggest that $\sim$ 70% of the energy density in the universe has the properties of a cosmological constant with $\Lambda \sim (10^{-3}\ \mathrm{eV})^4$. This is much smaller than other scales in particle physics. In particular, the Standard Model is valid at least up to 10^{12} eV. Why don't the zero point energies of these fields contribute to Λ? Or, if they do, what cancels them to such high accuracy? This is the cosmological constant problem. No one knows the answer!

In fact, if QFT is a good description of the universe down to the Planck scale, we would expect a cosmological constant of the order of M_{pl}^4. This is about 10^{120} times greater than the measured value !. This discrepancy has been called "the worst theoretical prediction in the history of physics", and is still unresolved.

3.2. The Casimir Effect (1948)

Using the normal ordering prescription, we can happily set $E_0 = 0$, while chanting the mantra that only energy differences can be measured. But we should be careful, for there is a situation where differences in the energy of vacuum fluctuations themselves can be measured.

The typical example is of two uncharged metallic plates in a vacuum, placed a few micrometers apart (see Figure 1). In a classical description, the lack of an external field also means that there is no field between the plates, and no force would be measured between them. However, in the framework of QFT, fluctuations in the vacuum energy would cause a force to act between the plates. This force was first predicted by Hendrik Casimir in 1948, and is named after him. It was experimentally observed in 1958.

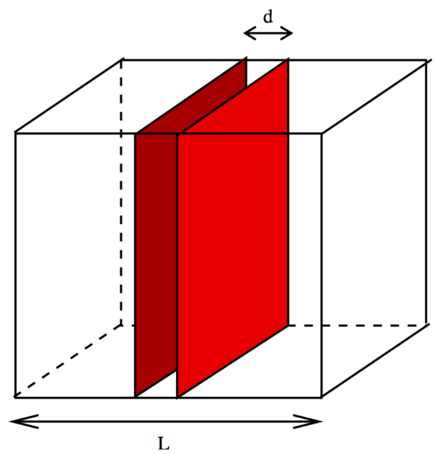

Fig. 1.— Casimir Effect. Two uncharged metallic plates are placed nearby. Classically, lack of fields imply that no force between them is expected. However, in QFT, vacuum fluctuations imply that a force must act between the plates.

To regulate the infra-red divergences, we will make the x^1 direction periodic, with size L, and impose periodic boundary conditions such that

$$\phi(\vec{x}) = \phi(\vec{x} + L\vec{n}), \tag{31}$$

with $\vec{n} = (1, 0, 0)$. We will leave y and z untouched, but remember that we should compute all physical quantities per unit area A. We insert two reflecting plates, separated by a distance $d \ll L$ in the x^1 direction. Being reflective, the plates impose the boundary condition $\phi(x) = 0$ at their position. The presence of these plates affects the Fourier decomposition of the field and, in particular, means that the momentum of the field inside the plates is quantized as

$$\vec{p} = \left(\frac{n\pi}{d}, p_y, p_z\right) \quad n = 1, 2, 3, ... \tag{32}$$

For a *massless* scalar field, $\omega_{\vec{p}} = \vec{p}$, and the ground state energy between the plates is (see Equation 26),

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dp_y dp_z}{(2\pi)^2} \frac{1}{2} \sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2}, \tag{33}$$

while the energy *outside* the plates is $E(L - d)$. The total energy is therefore

$$E_{tot} = E(d) + E(L - d) \tag{34}$$

which - at least naively - depends on d. If this naive guess is true, it would mean that there is a force on the plates due to the fluctuations of the vacuum. This is the **Casimir force**.

In the real world, the effect is due to the vacuum fluctuations of the electromagnetic field, with the boundary conditions imposed by conducting plates. Here we model this effect with a scalar.

But there is a problem. E is infinite! What to do? The problem comes from the arbitrarily high momentum modes. We could regulate this in a number of different ways. Physically one could argue that any real plate cannot reflect waves of arbitrarily high frequency: at some point, things begin to leak. Mathematically, we want to find a way to neglect modes of momentum $p \gg a^{-1}$ for some distance scale $a \ll d$, known as the ultra-violet (UV) cut-off.

One way to do this is to change the integral in Equation 33 to

$$\frac{E(d)}{A} = \sum_{n=1}^{\infty} \int \frac{dp_y dp_z}{(2\pi)^2} \frac{1}{2} \left(\sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2} \right) e^{-a\sqrt{\left(\frac{n\pi}{d}\right)^2 + p_y^2 + p_z^2}}, \tag{35}$$

which has the property that as $a \to 0$, we regain the full, infinite, expression in Equation 33. However Equation 35 is finite, and gives us something we can easily work with. Of course, we made it finite in a rather ad-hoc manner and we better make sure that any physical quantity we calculate doesn't depend on the UV cut-off a, otherwise it's not something we can really trust.

The integral in Equation 35 is doable, although a bit complicated. It is much simpler to look at the problem in $d = 1 + 1$ dimensions, rather than the full $d = 3 + 1$ dimensions. We will find that all the same physics is at play. In $1 + 1$ dimensions, the energy is given by

$$E_{1+1}(d) = \frac{\pi}{2d} \sum_{n=1}^{\infty} n. \tag{36}$$

We now regulate this sum by introducing the UV cutoff a that have introduced above. This renders the expression finite, allowing us to start manipulating it. Thus,

$$\begin{aligned} E_{1+1}(d) &\to \tfrac{\pi}{2d} \textstyle\sum_{n=1}^{\infty} n e^{-an\pi/d} \\ &= -\tfrac{1}{2}\tfrac{\partial}{\partial a} \textstyle\sum_{n=1}^{\infty} e^{-an\pi/d} \\ &= -\tfrac{1}{2}\tfrac{\partial}{\partial a} \tfrac{1}{1-e^{-a\pi/d}} \\ &= \tfrac{\pi}{2d} \tfrac{e^{a\pi/d}}{\left(e^{a\pi/d}-1\right)^2} \\ &= \tfrac{d}{2\pi a^2} - \tfrac{\pi}{24d} + \mathcal{O}(a^2), \end{aligned} \tag{37}$$

where in the last line we used the fact that $a \ll d$. We can now compute the full energy,

$$E_{1+1} = E_{1+1}(d) + E_{1+1}(L-d) = \frac{L}{2\pi a^2} - \frac{\pi}{24}\left(\frac{1}{d} + \frac{1}{L-d}\right) + \mathcal{O}(a^2). \tag{38}$$

This is still infinite in the limit $a \to 0$, which is to be expected. However, the **force** between the plates is given by

$$\frac{\partial E_{1+1}}{\partial d} = \frac{\pi}{24 d^2} + ... \tag{39}$$

where the ... include terms of the size d/L and a/d. The key point is that as we remove both the regulators, and take the limits $a \to 0$ and $L \to \infty$, the force between the plates remains finite. This is the Casimir force.

If we ploughed through the analogous calculation in $d = 3+1$ dimensions, and performed the integral in Equation 35, we would find the result

$$\frac{1}{A}\frac{\partial E}{\partial d} = \frac{\pi^2}{480 d^4}. \tag{40}$$

The true Casimir force is twice as large as this, due to the two polarization states of the photon.

4. Recovering Particles

Following our discussion on the vacuum, let us now turn our attention to excitations of the field. It is easy to verify that

$$\left[H, a^\dagger_{\vec{p}}\right] = \omega_{\vec{p}} a^\dagger_{\vec{p}} \quad \text{and} \quad [H, a_{\vec{p}}] = -\omega_{\vec{p}} a_{\vec{p}} \tag{41}$$

which means that, just as for the harmonic oscillator, we can construct energy eigenstates by acting on the vacuum $|0\rangle$ with $a^\dagger$. Let

$$|\vec{p}\rangle = a^\dagger_{\vec{p}}|0\rangle. \tag{42}$$

This state has energy

$$H|\vec{p}\rangle = \omega_{\vec{p}}|\vec{p}\rangle \quad \text{with} \quad \omega^2_{\vec{p}} = \vec{p}^2 + m^2 \tag{43}$$

We recognize this as the relativistic dispersion relation for a particle of mass m and 3-momentum $\vec{p}$,

$$E^2_{\vec{p}} = \vec{p}^2 + m^2. \tag{44}$$

This motivates us to interpret the state $|\vec{p}\rangle$ as the momentum eigenstate of a single particle of mass m. To stress this, from now on we will write $E_{\vec{p}}$ everywhere instead of $\omega_{\vec{p}}$.

Let's check this particle interpretation by studying the other quantum numbers of $|\vec{p}\rangle$. We may take the classical total momentum $\vec{P}$ introduced in "classical fields" (Equation 46:

$P^i = \int d^3x T^{0i} = \int d^3x \dot{\phi}\partial^\mu\phi)$ **and turn it into an operator**. After normal ordering, it becomes

$$\vec{P} = -\int d^3x : \pi\vec{\nabla}\phi := \int \frac{d^3p}{(2\pi)^3}\vec{p}a^\dagger_{\vec{p}}a_{\vec{p}} \tag{45}$$

where the minus sign in the first equation arises from lowering indices ($\partial_\mu\phi = g_{\mu\nu}\partial^\nu\phi$), and the last equality is obtained in a similar way to the Hamiltonian in Equation 29.

The result in Equation 45 makes it immediate to see that acting on the state $|\vec{p}\rangle$ with the operator $\vec{P}$, we find that it is indeed an eigenstate, with momentum $\vec{p}$:

$$\vec{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle. \tag{46}$$

The state $|\vec{p}\rangle$ thus have both the expected energy and the expected momentum of a particle !.

Another property of $|\vec{p}\rangle$ that we can study is its *intrinsic* angular momentum (=**spin**). Once again, we may take the classical expression for the total angular momentum of the field (given in "classical fields", Equation 55) and turn it into an operator:

$$J^i = \epsilon^{ijk}\int d^3x(\tilde{j}^0)^{jk} \tag{47}$$

It is not difficult to show that acting on the one-particle state with zero momentum, $J^i|\vec{p} = \vec{0}\rangle = 0$ which we interpret as telling us that the particle carries no internal angular momentum. In other words, **quantizing a scalar field gives rise to a spin 0 particle**.

Note the following: (I) the state $|\vec{p} = \vec{0}\rangle$ is NOT similar to the state $|0\rangle$: the former is a state of an existing particle, which happens to be at rest (zero linear momentum). (II) We obtained the spin of a particle by applying the angular momentum operator to a particle at rest; this is as opposed to QM where the spin is introduced independently of the external angular momentum.

4.1. Multi-Particle States, Bosonic Statistics and Fock Space

We have just shown that when acting on a vacuum state, the operator $a^\dagger_{\vec{p}}$ creates a momentum $\vec{p}$ and energy $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. We thus called this excitation a **particle**, being a discrete entity that have the proper relativistic energy-momentum relation.

We can now proceed and create multi-particle states by acting multiple times with $a^\dagger$'s. We interpret the state in which $n - a^\dagger$'s act on the vacuum as an n**-particle state**,

$$|\vec{p}_1, ..., \vec{p}_n\rangle = a^\dagger_{\vec{p}_1}...a^\dagger_{\vec{p}_n}|0\rangle. \tag{48}$$

Because all the $a^\dagger$'s *commute* among themselves, the state is symmetric under exchange of any two particles. For example,

$$|\vec{p},\vec{q}\rangle = |\vec{q},\vec{p}\rangle \tag{49}$$

This means that the particles are **bosons**. Note that the theory automatically gave us spin-0 for the bosons.

The full Hilbert space of our theory is spanned by acting on the vacuum with all possible combinations of $a^\dagger$'s,

$$|0\rangle,\ a^\dagger_{\vec{p}}|0\rangle,\ a^\dagger_{\vec{p}}a^\dagger_{\vec{q}}|0\rangle,\ a^\dagger_{\vec{p}}a^\dagger_{\vec{q}}a^\dagger_{\vec{r}}|0\rangle, ... \tag{50}$$

This space is known as **Fock space**. The Fock space is simply the sum of the n-particle Hilbert spaces, for all $n \geq 0$.

There is a useful operator which counts the number of particles in a given state in the Fock space. It is called the **number operator** N,

$$N = \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} a_{\vec{p}}, \tag{51}$$

and satisfies $N|\vec{p}_1, ..., \vec{p}_n\rangle = n|\vec{p}_1, ..., \vec{p}_n\rangle$. The number operator commutes with the Hamiltonian, $[N, H] = 0$, ensuring that particle number is conserved. This means that we can place ourselves in the n-particle sector, and stay there. This is a property of free theories, but will no longer be true when we consider interactions: interactions create and destroy particles, taking us between the different sectors in the Fock space.

4.2. Operator Values Distributions

Although we are referring to the states $|\vec{p}\rangle$ as particles, note that they are not localized in space in any way - they are **momentum eigenstates**. Recall that in quantum mechanics the position and momentum eigenstates are **not** good elements of the Hilbert space since they are not normalizable (they normalize to delta-functions). Similarly, in quantum field theory neither the operators $\phi(\vec{x})$, nor $a_{\vec{p}}$ are good operators acting on the Fock space. This is because they don't produce normalizable states. For example,

$$\langle 0|a_{\vec{p}}a^\dagger_{\vec{p}}|0\rangle = \langle\vec{p}|\vec{p}\rangle = (2\pi)^3\delta(0) \quad \text{and} \quad \langle 0|\phi(\vec{x})\phi(\vec{x})|0\rangle = \langle\vec{x}|\vec{x}\rangle = \delta(0), \tag{52}$$

where we have assumed the normalization $\langle 0|0\rangle = 1$, and the commutation relations in Equation 20. Technically, these are **operator valued distributions**, rather than functions. This means that although $\phi(\vec{x})$ has a well defined vacuum expectation value, $\langle 0|\phi(\vec{x})|0\rangle = 0$, the fluctuations of the operator at a fixed point are infinite, $\langle 0|\phi(\vec{x})\phi(\vec{x})|0\rangle = \infty$.

We can construct well defined operators by smearing these distributions over space. For example, we can create a wavepacket

$$|\phi\rangle = \int \frac{d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \phi(\vec{p})|\vec{p}\rangle \tag{53}$$

which is partially localized in both position and momentum space. (A typical state might be described by the Gaussian $\phi(\vec{p}) = exp(-\vec{p}^2/2m^2)$).

4.3. Relativistic Normalization

We have defined the vacuum $|0\rangle$, which we normalized as $\langle 0|0\rangle = 1$. The 1-particle states $|\vec{p}\rangle = a^\dagger_{\vec{p}}|0\rangle$ then satisfy:

$$\langle\vec{p}|\vec{q}\rangle = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}). \tag{54}$$

But now we can ask: is this Lorentz invariant ? The problem of course is that we have only 3-vectors.

Suppose we have a Lorentz transformation:

$$p^\mu \rightarrow (p')^\mu = \Lambda^\mu{}_\nu p^\nu, \tag{55}$$

such that the 3-vector transforms as $\vec{p} \rightarrow \vec{p}'$. In the quantum theory, it would be preferable if the two states are related by a unitary transformation,

$$|\vec{p}\rangle \rightarrow |\vec{p}'\rangle = U(\Lambda)|\vec{p}\rangle. \tag{56}$$

In such a case, the normalizations of $|\vec{p}\rangle$ and $|\vec{p}'\rangle$ are the same whenever $\vec{p}$ and $\vec{p}'$ are related by a Lorentz transformation.

However, we have not been careful with the normalization: in general, we could get

$$|\vec{p}\rangle \rightarrow \lambda(\vec{p},\vec{p}')|\vec{p}'\rangle, \tag{57}$$

for some unknown function $\lambda(\vec{p},\vec{p}')$.

How do we figure that out ? The trick is to look at an object which we know is Lorentz invariant. One such object is the identity operator on one-particle states (which is really the projection operator onto one-particle states). With the normalization given in Equation 54 we know this is given by

$$1 = \int \frac{d^3p}{(2\pi)^3}|\vec{p}\rangle\langle\vec{p}|. \tag{58}$$

This operator is Lorentz invariant; however, it consists of two terms: the measure $\int d^3p$ and the projector $|\vec{p}\rangle\langle\vec{p}|$. These terms are **not** individually Lorentz invariant.

Claim. The Lorentz invariant measure is

$$\int \frac{d^3p}{2E_{\vec{p}}}. \tag{59}$$

Proof. $\int d^4p$ is obviously Lorentz invariant. And the relativistic dispersion relation for a massive particle,

$$p_\mu p^\mu = m^2 \quad \Rightarrow \quad p_0^2 = E_{\vec{p}}^2 = \vec{p}^2 + m^2, \tag{60}$$

is also Lorentz invariant. Solving for p_0, there are two branches of solutions: $p_0 = \pm E_{\vec{p}}$. But the choice of branch is another Lorentz invariant concept. So piecing everything together, the following combination must be Lorentz invariant,

$$\int d^4p\,\delta(p_0^2 - \vec{p}^2 - m^2)\bigg|_{p_0>0} = \int \frac{d^3p}{2p_0}\bigg|_{p_0=E_{\vec{p}}}, \tag{61}$$

which completes the proof. (In the last line we used the properties of the delta function, $\delta(f(x)) = \delta(x)/|f'(x)|$).

From this result we can figure out everything else. For example, the Lorentz invariant δ-function for 3-vectors is

$$2E_{\vec{p}}\delta^{(3)}(\vec{p}-\vec{q}), \tag{62}$$

which follows because

$$\int \frac{d^3p}{2E_{\vec{p}}} 2E_{\vec{p}}\delta^{(3)}(\vec{p}-\vec{q}) = 1. \tag{63}$$

Thus, we finally learn that the relativistically normalized momentum states are given by

$$|p\rangle = \sqrt{2E_{\vec{p}}}|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}a_{\vec{p}}^\dagger|0\rangle \tag{64}$$

Notice that our notation is rather subtle: the relativistically normalized momentum state $|p\rangle$ differs from $|\vec{p}\rangle$ just by the factor $\sqrt{2E_{\vec{p}}}$. These states now satisfy

$$\langle p|q\rangle = (2\pi)^3 2E_{\vec{p}}\delta^{(3)}(\vec{p}-\vec{q}). \tag{65}$$

Finally, we can rewrite the identity on one-particle states as

$$1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}|p\rangle\langle p|. \tag{66}$$

Note that some textbooks also define relativistically normalized creation operators by $a^\dagger(p) = \sqrt{2E_{\vec{p}}}a_{\vec{p}}^\dagger$. We will not use this notation here.

5. Complex Scalar Fields

Let us now consider a complex scalar field $\psi(x)$ with the Lagrangian

$$\mathcal{L} = \partial_\mu \psi^\star \partial^\mu \psi - M^2 \psi^\star \psi. \tag{67}$$

Note that, as opposed to the Lagrangian for the real scalar field ($\mathcal{L} = (1/2)\partial_\mu \partial^\mu \phi - (1/2)m^2\phi^2$: see "Classical Fields", Equation 7), there is no factor of (1/2) in front of the Lagrangian of the complex scalar field. If we write ψ in terms of real scalar fields by $\psi = (\phi_1 + i\phi_2)/\sqrt{2}$, we automatically get the factor (1/2) coming from the ($1/\sqrt{2}$'s).

The equations of motion are

$$\begin{aligned} \partial_\mu \partial^\mu \psi + M^2 \psi = 0 \\ \partial_\mu \partial^\mu \psi^\star + M^2 \psi^\star = 0, \end{aligned} \tag{68}$$

where the second equation is the complex conjugate of the first. We expand the complex field operator as a sum of plane waves as

$$\begin{aligned} \psi(x) &= \int \tfrac{d^3p}{(2\pi)^3} \tfrac{1}{\sqrt{2E_{\vec{p}}}} \left(b_{\vec{p}} e^{+i\vec{p}\cdot\vec{x}} + c^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right), \\ \psi^\dagger(x) &= \int \tfrac{d^3p}{(2\pi)^3} \tfrac{1}{\sqrt{2E_{\vec{p}}}} \left(b^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} + c_{\vec{p}} e^{+i\vec{p}\cdot\vec{x}} \right). \end{aligned} \tag{69}$$

Since the classical field ψ is not real, the corresponding quantum field ψ is not hermitian. This is the reason that we have different operators b and $c^\dagger$ appearing in the positive and negative frequency parts. ($\psi^\dagger$ will be the quantum operator associated with the real part of the field $\psi^\star$.)

The classical field momentum is $\pi = \partial\mathcal{L}/\partial\dot{\psi} = \dot{\psi}^\star$. We also turn this into a quantum operator field, which we write as

$$\begin{aligned} \pi &= \int \tfrac{d^3p}{(2\pi)^3} i \sqrt{\tfrac{E_{\vec{p}}}{2}} \left(b^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} - c_{\vec{p}} e^{+i\vec{p}\cdot\vec{x}} \right), \\ \pi^\dagger &= \int \tfrac{d^3p}{(2\pi)^3} (-i) \sqrt{\tfrac{E_{\vec{p}}}{2}} \left(b_{\vec{p}} e^{+i\vec{p}\cdot\vec{x}} - c^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \right). \end{aligned} \tag{70}$$

Similarly to Equations 18 and 19, we can think of Equations 69, 70 as defining the operators $b_{\vec{p}}$, $b^\dagger_{\vec{p}}$, $c_{\vec{p}}$ and $C^\dagger_{\vec{p}}$. Equation 70 is different than the definition of π, which we used when treating classical fields.

The commutation relations between fields and momenta are given by

$$[\psi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}) \quad \text{and} \quad \left[\psi(\vec{x}), \pi^\dagger(\vec{y})\right] = 0, \tag{71}$$

together with the other commutation relations, related by complex conjugation, as well as the usual $[\psi(\vec{x}), \psi(\vec{y})] = \left[\psi(\vec{x}), \psi^\dagger(\vec{y})\right] = 0$, etc. One can easily check that these field commutation relations are equivalent to the commutation relations for the operators $b_{\vec{p}}$ and $c_{\vec{p}}$,

$$\begin{aligned}\left[b_{\vec{p}}, b_{\vec{q}}^{\dagger}\right] &= (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}),\\ \left[c_{\vec{p}}, c_{\vec{q}}^{\dagger}\right] &= (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q}),\end{aligned} \tag{72}$$

and

$$[b_{\vec{p}}, b_{\vec{q}}] = [c_{\vec{p}}, c_{\vec{q}}] = [b_{\vec{p}}, c_{\vec{q}}] = \left[b_{\vec{p}}, c_{\vec{q}}^{\dagger}\right] = 0. \tag{73}$$

In summary, quantizing a complex scalar field gives rise to **two** creation operators, $b_{\vec{p}}^{\dagger}$ and $c_{\vec{p}}^{\dagger}$. These have the interpretation of creating *two types of particle*, **both of mass** M **and both spin zero**. *They are interpreted as particles and anti-particles.* In contrast, for a real scalar field there is only a single type of particle: for a real scalar field, the particle is its own antiparticle.

Recall that the theory described by the Lagrangian in Equation 67 has a classical conserved charge (see "Classical Fields", Equation 63):

$$Q = i\int d^3x\left(\dot{\psi}^{\star}\psi - \psi^{\star}\dot{\psi}\right) = i\int d^3x\left(\pi\psi - \psi^{\star}\pi^{\star}\right). \tag{74}$$

After normal ordering, this becomes the quantum operator:

$$Q = \int\frac{d^3p}{(2\pi)^3}\left(c_{\vec{p}}^{\dagger}c_{\vec{p}} - b_{\vec{p}}^{\dagger}b_{\vec{p}}\right) = N_c - N_b. \tag{75}$$

We thus find that Q counts the number of c particles, minus the number of b particles. We will interpret $c^{\dagger}$ as creating anti-particles, and thus Q counts the number of anti-particles (created by $c^{\dagger}$), minus the number of particles (created by $b^{\dagger}$). Furthermore, we have $[H, Q] = 0$, ensuring that Q is a conserved quantity in quantum theory.

Of course, in our free field theory this isn't such a big deal, because both N_c and N_b are separately conserved. However, we will soon see that in interacting theories, as long as they obey the phase symmetry $\psi \to e^{i\alpha}\psi$, Q survives as a conserved quantity, while N_c and N_b individually do not.

6. The Heisenberg Picture

At the moment, our operators $\phi(\vec{x})$ and $\pi(\vec{x})$, depend on *space*, but not on *time*. Meanwhile, the one-particle states evolve in time in accordance to Schrödinger's equation,

$$i\frac{d|\vec{p}(t)\rangle}{dt} = H|\vec{p}(t)\rangle \quad\Rightarrow\quad |\vec{p}(t)\rangle = e^{-iE_{\vec{p}}t}|\vec{p}(t=0)\rangle. \tag{76}$$

This can potentially cause a trouble: although we started with a Lorentz invariant Lagrangian, we slowly butchered it as we quantized, introducing a preferred time coordinate t. It's not at all obvious that the theory is still Lorentz invariant after quantization!.

We can change this by going to the **Heisenberg picture**, where time dependence is assigned to the operators $\mathcal{O}$,

$$\mathcal{O}_H = e^{iHt}\mathcal{O}_S e^{-iHt}, \tag{77}$$

(Equation 77 is the definition of the operators in the Heisenberg picture).

The equation of motion obeyed by the Heisenberg operator is

$$\frac{d\mathcal{O}_H}{dt} = i\left[H, \mathcal{O}_H\right] \tag{78}$$

where the subscripts S and H tell us whether the operator is in the Schrödinger or Heisenberg picture. In field theory, we drop these subscripts and we will denote the picture by specifying whether the fields depend on space $\phi(\vec{x})$ (the Schrödinger picture) or spacetime $\phi(\vec{x}, t) = \phi(x)$ (the Heisenberg picture).

The operators in the two pictures agree at a fixed time, say, $t = 0$. The commutation relations (Equation 2) become equal time commutation relations in the Heisenberg picture,

$$\begin{aligned} [\phi(\vec{x}, t), \phi(\vec{y}, t)] &= [\pi(\vec{x}, t), \pi(\vec{y}, t)] = 0 \\ [\phi(\vec{x}, t), \pi(\vec{y}, t)] &= i\delta^{(3)}(\vec{x} - \vec{y}). \end{aligned} \tag{79}$$

Working in the Heisenberg picture where the operator $\phi(x) = \phi(\vec{x}, t)$ depends on time, we can study how it evolves. For example, we have

$$\begin{aligned} \dot{\phi} = i\left[H, \phi\right] &= \frac{i}{2}\left[\int d^3y \left(\pi(y)^2 + \left(\nabla\phi(y)\right)^2 + m^2\phi^2(y)\right), \phi(x)\right] \\ &= i\int d^3y \pi(y)(-i)\delta^{(3)}(\vec{y} - \vec{x}) = \pi(x) \end{aligned} \tag{80}$$

Meanwhile, the equation of motion for π reads,

$$\begin{aligned} \dot{\pi} = i\left[H, \pi\right] &= \frac{i}{2}\left[\int d^3y \left(\pi(y)^2 + \left(\nabla\phi(y)\right)^2 + m^2\phi^2(y)\right), \pi(x)\right] \\ &= \frac{i}{2}\int d^3y \left(\nabla\phi(y)\left[\nabla\phi(y), \pi(x)\right] + \left[\nabla\phi(y), \pi(x)\right]\nabla\phi(y) + 2im^2\phi(y)\delta^{(3)}(\vec{y} - \vec{x})\right) \\ &= \frac{i}{2}\int d^3y \left(\nabla\phi(y)\nabla_{(y)}\left[\phi(y), \pi(x)\right] + \nabla_{(y)}\left[\phi(y), \pi(x)\right]\nabla\phi(y) + 2im^2\phi(y)\delta^{(3)}(\vec{y} - \vec{x})\right) \\ &= -\left(\int d^3y \nabla_{(y)}\delta^{(3)}(\vec{x} - \vec{y})\nabla\phi(y)\right) - m^2\phi(x) \\ &= \nabla^2\phi - m^2\phi, \end{aligned} \tag{81}$$

where in the third line I have introduced the subscript (y) on $\nabla_{(y)}$ to remind the argument of the derivative; We further used the commutations relations in Equation 79. In the last line, we have used the property of the derivative of the δ-function, $\int f(x)\delta'(x - x')dx = -f'(x')$, which is obtained by performing integration by parts.

Using now Equation 81 inside equation 80, we find that by simply moving to the Heisenberg picture, the field operator ϕ satisfies the Klein-Gordon equation,

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0. \tag{82}$$

Things thus begin to look more relativistic. We can write the Fourier expansion of $\phi(x)$ by starting from the Fourier transform of $\phi(\vec{x})$ in Equation 18, use the definition of Heisenberg operators in Equation 77 and noting that

$$e^{iHt} a_{\vec{p}} e^{-iHt} = e^{-iE_{\vec{p}}t} a_{\vec{p}} \;\; \text{and} \;\; e^{iHt} a^\dagger_{\vec{p}} e^{-iHt} = e^{+iE_{\vec{p}}t} a^\dagger_{\vec{p}} \tag{83}$$

which follows from the commutation relations, $[H, a_{\vec{p}}] = -E_{\vec{p}} a_{\vec{p}}$, and $\left[H, a^\dagger_{\vec{p}}\right] = +E_{\vec{p}} a^\dagger_{\vec{p}}$. This then gives

$$\phi(x) = \phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip\cdot x} + a^\dagger_{\vec{p}} e^{+ip\cdot x} \right). \tag{84}$$

This looks very similar to the expansion in Equation 18, except that the exponent is now written in terms of 4-vectors, $p \cdot x = E_{\vec{p}} t - \vec{p} \cdot \vec{x}$. Note also that the sign has flipped in the exponent, due to Minkowski metric contraction. It is simple to check that Equation 84 indeed satisfies the Klein-Gordon Equation 82.

6.1. Commutation Relations and Causality

The basic principle requirement is that **nothing travels faster than light**. So far, we have shown that $\phi(x)$ in the Heisenberg picture satisfies the Klein-Gordon equation. However, we based our calculation on that ϕ and π satisfy **equal time** commutation relations (see Equation 79),

$$[\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y}). \tag{85}$$

However, what about operators at *arbitrary* space-time separations? In particular, for our theory to be **causal**, we must require that *all spacelike separated operators commute*: namely, an operator $\mathcal{O}_1$ should commute with all operators $\mathcal{O}_2$ which are outside the light cone (see Figure 2). Thus,

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0 \quad \forall\ (x-y)^2 < 0 \tag{86}$$

This ensures that a measurement at x cannot affect a measurement at y when x and y are not causally connected (namely, outside the light cone).

Does our theory satisfy this crucial property ? Let us define:

$$\Delta(x - y) \equiv [\phi(x), \phi(y)]. \tag{87}$$

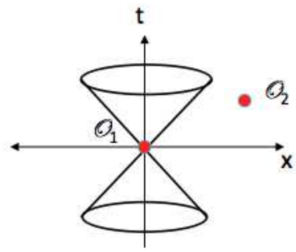

Fig. 2.— To ensure causality, an operator should commute with all operators outside the lightcone.

The objects on the right-hand side of this expression are operators. However, it's easy to check by direct substitution that the left-hand side is simply a c-number function with the integral expression:

$$\Delta(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) . \tag{88}$$

what do we know about this function?

- It is Lorentz invariant: this is insured by the appearance of the Lorentz invariant measure $\int d^3p/2E_{\vec{p}}$ introduced in Equation 59, and the 4-vectors $p \cdot (x-y)$.
- It does not vanish for timelike separation. For example, taking $x - y = (t, 0, 0, 0)$ gives $[\phi(\vec{x}, 0), \phi(\vec{x}, t)] \sim e^{-imt} - e^{+imt}$. (Recall that $E_{\vec{p}} = m$).
- It does vanish for space-like separations. This follows by noting that $\Delta(x-y) = 0$ at equal times for all $(x-y)^2 = -(\vec{x}-\vec{y})^2 < 0$, which we can see implicitly by writing

$$[\phi(\vec{x}, t), \phi(\vec{y}, t)] = \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{\vec{p}^2 + m^2}} \left(e^{i\vec{p}\cdot(\vec{x}-\vec{y})} - e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right) \tag{89}$$

  and noticing that we can flip the sign of $\vec{p}$ in the last exponent as it is an integration variable. Moreover, since $\Delta(x-y)$ is Lorentz invariant, it can only depend on $(x-y)^2$, and must therefore vanish for all $(x-y)^2 < 0$.

We therefore learn that our theory is indeed causal with commutators vanishing outside the lightcone; **the theory is indeed Lorentz invariant**. This property will continue to hold in interacting theories; indeed, it is usually given as one of the axioms of local quantum field theories. I should mention however that the fact that $[\phi(x), \phi(y)]$ is a c-number function, rather than an operator, is a property of free fields only.

7. Propagators

In order to ensure Lorentz invariance, we required two operators outside the lightcone to commute. We saw that they do; thus, two measurements carried outside the light cone do not affect each other.

We could ask a different question to probe the causal structure of the theory. Prepare a particle at spacetime point y. **What is the amplitude $\langle 0|\phi(x)\phi(y)|0\rangle$ to find it at point x?** We can calculate this:

$$\begin{aligned}\langle 0|\phi(x)\phi(y)|0\rangle &= \int \frac{d^3p d^3p'}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{p}'}}}\langle 0|a_{\vec{p}}a^\dagger_{\vec{p}'}|0\rangle e^{-ip\cdot x+ip'\cdot y} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip\cdot(x-y)} \equiv D(x-y).\end{aligned} \tag{90}$$

(where we have used the fact that $a_{\vec{p}}|0\rangle = 0$ and $\langle 0|a^\dagger_{\vec{p}} = 0$). The function $D(x-y)$ is called the **propagator**.

For spacelike separations (outside the lightcone), $(x-y)^2 < 0$, one can show that $D(x-y)$ decays like

$$D(x-y) \sim e^{-m|\vec{x}-\vec{y}|} \neq 0 \tag{91}$$

So it decays exponentially quickly outside the lightcone but, nonetheless, is **non-vanishing!** The quantum field appears to leak out of the lightcone.

Yet we have just seen that space-like measurements commute and the theory is causal. How do we reconcile these two facts? We can rewrite the calculation in Equation 89 as

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x) = 0 \quad \text{if} \quad (x-y)^2 < 0. \tag{92}$$

There are words you can drape around this calculation. When $(x-y)^2 < 0$, there is no Lorentz invariant way to order events. If a particle can travel in a spacelike direction from $x \to y$, it can just as easily travel from $y \to x$. In any measurement, the amplitudes for these two events cancel.

With a complex scalar field, it is more interesting. We can look at the equation $\left[\psi(x), \psi^\dagger(y)\right] = 0$ outside the lightcone. The interpretation now is that the amplitude for the particle to propagate from $x \to y$ cancels the amplitude for the antiparticle to travel from $y \to x$. In fact, this interpretation is also there for a real scalar field because the particle is its own antiparticle.

7.1. The Feynman Propagator

As we will see shortly, one of the most important quantities in interacting field theory is the **Feynman propagator**, defined by

$$\Delta_F(x-y) = \langle T\phi(x)\phi(y)|0\rangle = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & y^0 > x^0 \end{cases} \tag{93}$$

where T stands for **time ordering**, placing all operators evaluated at later times to the left:

$$T\phi(x)\phi(y) \equiv \begin{cases} \phi(x)\phi(y) & x^0 > y^0 \\ \phi(y)\phi(x) & y^0 > x^0 \end{cases} \tag{94}$$

Claim. The Feynman propagator can be written in terms of a 4-momentum integral as

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}. \tag{95}$$

Notice that this is the *first time in this course that we've integrated over 4-momentum.* Until now, we integrated only over 3-momentum, with p^0 fixed by the **mass-shell condition** to be $p^0 = E_{\vec{p}}$. In contrast, in Equation 95 for Δ_F, there is no such condition, as we are integrating over p^0.

Note, however, that as it stands, the integral in Equation 95 is **ill-defined**: this is because, for each (fixed) value of $\vec{p}$, the denominator $p^2 - m^2 = (p^0)^2 - \vec{p}^2 - m^2$ **produces a pole** when $p^0 = \pm E_{\vec{p}} = \pm\sqrt{\vec{p}^2 + m^2}$. We need a prescription for avoiding these singularities in the p^0 integral.

The solution is to define the integral by a contour on the complex p^0 plane (pretending p^0 is a complex number), and integrate over the real axis: only that there are two points in which the integral diverges. We thus chose to perform the integral along the contour shown in Figure 3.

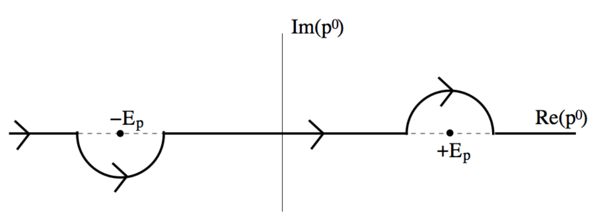

Fig. 3.— The contour of integration for the Feynman propagator.

Proof. We are going to use the **residue theorem** (see "preliminaries"). First, we write

$$\frac{1}{p^2-m^2} = \frac{1}{(p^0)^2 - E_{\vec{p}}^2} = \frac{1}{(p^0-E_{\vec{p}})(p^0+E_{\vec{p}})}. \tag{96}$$

Thus, the residue of the pole at $p^0 = \pm E_{\vec{p}}$ is $\pm 1/(2E_{\vec{p}})$.

In order to use the residue theorem, we need to close the contour of integration - either from above or from below. When $x^0 > y^0$, we close the contour on the **lower half plane**, where $p^0 \to -i\infty$. This ensures that the integrand vanishes, since for $x^0 > y^0$, the exponent $e^{-ip^0\cdot(x^0-y^0)} \to e^{-\infty} = 0$. Thus, the result we obtain using the residue theorem (which can be used only for closed contours) is the sum of the results of the original integral + zero.

By this choice of contour, the integral over p^0 picks up the residue at $p^0 = +E_{\vec{p}}$, which is equal to $-2\pi i/2E_{\vec{p}}$, where the minus sign arises because we took a clockwise contour (the residue itself is positive). Thus, when $x^0 > y^0$ we have

$$\begin{aligned} \Delta_F(x-y) &= \int \frac{d^3p}{(2\pi)^4}\frac{-2\pi i}{2E_{\vec{p}}} i e^{-iE_{\vec{p}}\cdot(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} = D(x-y), \end{aligned} \tag{97}$$

which is indeed the Feynman propagator for $x^0 > y^0$.

In contrast, when $y^0 > x^0$, we close the contour in an anti-clockwise direction in the **upper half plane** to get,

$$\begin{aligned} \Delta_F(x-y) &= \int \frac{d^3p}{(2\pi)^4}\frac{2\pi i}{-2E_{\vec{p}}} i e^{+iE_{\vec{p}}\cdot(x^0-y^0)+i\vec{p}\cdot(\vec{x}-\vec{y})} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} e^{-iE_{\vec{p}}\cdot(y^0-x^0)-i\vec{p}\cdot(\vec{y}-\vec{x})} \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} e^{-ip\cdot(y-x)} = D(y-x), \end{aligned} \tag{98}$$

where to go to from the second line to the third, we have flipped the sign of $\vec{p}$ which is valid since we integrate over d^3p and all other quantities depend only on $\vec{p}^2$. Once again we reproduce the Feynman propagator, which completed the proof.

Note. Instead of specifying the contour, we could write the Feynman propagator as

$$\Delta_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{ie^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon} \tag{99}$$

with infinitesimal $\epsilon > 0$. This has the effect of shifting the poles slightly off the real axis, so the integral along the real p^0 axis is equivalent to the contour shown in Figure 4. This way of writing the propagator is, for obvious reasons, called the "$i\epsilon$ prescription".

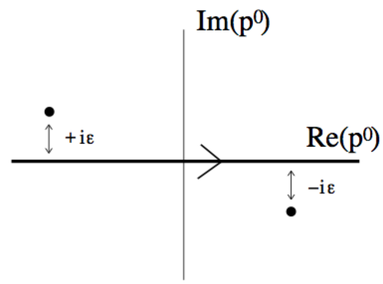

Fig. 4.— Alternative contour of integration for the Feynman propagator with the $i\epsilon$ prescription.

7.2. Green's Functions

There is another avatar of the propagator: **it is a Green's function for the Klein-Gordon operator**. If we stay away from the singularities, we have

$$\begin{aligned}(\partial_t{}^2 - \nabla^2 + m^2)\Delta_F(x-y) &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2}(-p^2+m^2)e^{-ip\cdot(x-y)} \\ &= -i\int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)} \\ &= -i\delta^{(4)}(x-y)\end{aligned} \tag{100}$$

Note that we did not make use of the contour anywhere in this derivation. Therefore, we could have picked different contours, which also give rise to Green's functions.

For example, the **retarded** Green's function $\Delta_R(x-y)$ is defined by the contour shown in Figure 5. For $x^0 > y^0$, we close the contour from below, picking up both poles; for $y^0 > x^0$, we close the contour from above, and obtain 0. Thus, the retarded Green's function has the property

$$\Delta_R(x-y) = \begin{cases} D(x-y) - D(y-x) & x^0 > y^0 \\ 0 & x^0 < y^0 \end{cases} \tag{101}$$

The retarded Green's function is useful in classical field theory if we know the initial value of some field configuration and want to figure out what it evolves into in the presence of a source, meaning that we want to know the solution to the inhomogeneous Klein-Gordon equation,

$$\partial_\mu\partial^\mu\phi + m^2\phi = J(x), \tag{102}$$

for some fixed background function $J(x)$.

Similarly, one can define the **advanced** Green's function $\Delta_A(x-y)$ which vanishes when $y^0 < x^0$ (see Figure 6), and is useful if we know the end point of a field configuration and

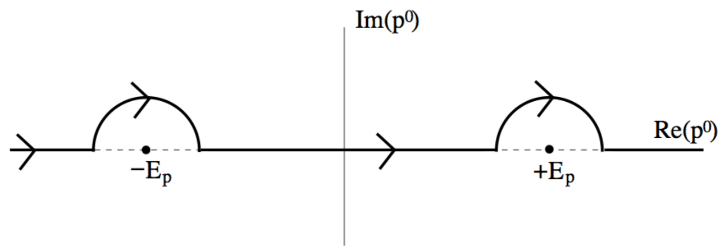

Fig. 5.— The retarded contour.

want to figure out where it came from. In the quantum theory, we will see that the Feynman Green's function is most relevant.

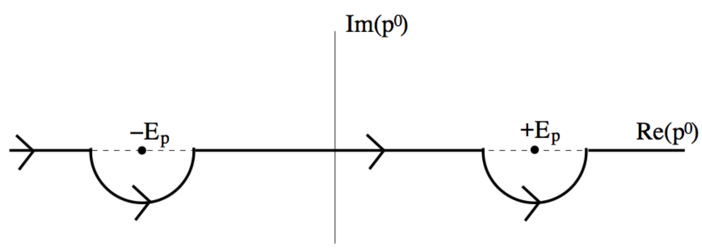

Fig. 6.— The advanced contour.

8. Non-Relativistic Fields

Let us now obtain the non-relativistic limit fields. In this limit, $E = m + |\vec{p}|$, with the kinetic energy $|\vec{p}| \ll m$. Look at classical complex scalar field obeying the Klein-Gordon equation. We will decompose the field as

$$\psi(\vec{x}, t) = e^{-imt}\tilde{\psi}(\vec{x}, t). \tag{103}$$

The Klein-Gordon equation reads:

$$\partial_t{}^2\psi - \nabla^2\psi + m^2\psi = e^{-imt}\left[\ddot{\tilde{\psi}} - 2im\dot{\tilde{\psi}} - \nabla^2\tilde{\psi}\right] = 0, \tag{104}$$

with the m^2 term canceled by the time derivatives.

The non-relativistic limit of a particle is $|\vec{p}| \ll m$. Let us look at what it does to the field.

Using a Fourier transform, $\tilde{\psi}(p) = \int d^4x e^{-ip\cdot x}\tilde{\psi}(x)$, Equation 104 becomes

$$E'^2 + 2mE' - |\vec{p}|^2 = 0,$$

where $E' = p^0$. Fourier transforming the Klein-Gordon Equation (for the field ψ) gives $E^2 - |\vec{p}|^2 - m^2 = 0$. The relation between E and E' is obtained using Equation 103, $E' = E - m$. The transformed KG equation can be written as $|\vec{p}^2| = (E-m)(E+m) \approx 2m(E-m)$, where the last equality holds for $|\vec{p}| \ll m$. We can thus write the condition $|\vec{p}| \ll m$ as $2m(E-m) \ll m^2$, or $E' = E - m \ll m$. Thus, we find that in the non-relativistic limit, $|\ddot{\tilde{\psi}}| \ll m|\dot{\tilde{\psi}}|$ in Equation 104. In this limit, the KG equation becomes

$$i\frac{\partial\tilde{\psi}}{\partial t} = -\frac{1}{2m}\nabla^2\tilde{\psi}. \tag{105}$$

This looks very similar to the Schrödinger equation for a non-relativistic free particle of mass m. Except it doesn't have any probability interpretation - it is simply a classical field evolving through an equation that is first order in time derivatives.

A Lagrangian that gives rise to Equation 105 can be derived from the relativistic Lagrangian (Equation 67) by taking again the limit $\partial_t\psi \ll m\psi$, and is

$$\mathcal{L} = +i\psi^\star\dot{\psi} - \frac{1}{2m}\nabla\psi^\star\nabla\psi. \tag{106}$$

This Lagrangian has a conserved current arising from the internal symmetry $\psi \to e^{i\alpha}\psi$. The current has time and space components:

$$j^\mu = \left(\psi^\star\psi, \frac{i}{2m}(\psi^\star\nabla\psi - \psi\nabla\psi^\star)\right) \tag{107}$$

To move to the Hamiltonian formalism we compute the momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\psi^\star. \tag{108}$$

This means that the momentum conjugate to ψ is $i\psi^\star$. The momentum does not depend on time derivatives at all! This looks a little disconcerting but it's fully consistent for a theory which is first order in time derivatives. In order to determine the full trajectory of the field, we need only specify ψ and $\psi^\star$ at time $t = 0$: no time derivatives on the initial slice are required.

When calculating the Hamiltonian, we note that due to the existence of a "$p\dot{q}$" term in the Lagrangian (instead of the more familiar "$(1/2)p\dot{q}$", the time derivatives drop out in the Hamiltonian. We get

$$\mathcal{H} = \frac{1}{2m}\nabla\psi^\star\nabla\psi. \tag{109}$$

To quantize, we impose (in the Schrödinger picture) the canonical commutation relations

$$\begin{aligned}[\psi(\vec{x}), \psi(\vec{y})] = \left[\psi^\dagger(\vec{x}), \psi^\dagger(\vec{y})\right] = 0 \\ \left[\psi(\vec{x}), \psi^\dagger(\vec{y})\right] = \delta^{(3)}(\vec{x} - \vec{y}).\end{aligned} \tag{110}$$

We may expand $\psi(\vec{x})$ as a Fourier transform,

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}}, \tag{111}$$

where the commutation relations in Equation 110 require

$$\left[a_{\vec{p}}, a^\dagger_{\vec{q}}\right] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}). \tag{112}$$

The vacuum satisfies $a_{\vec{p}}|0\rangle = 0$, and the excitations are $a^\dagger_{\vec{p}_1} ... a^\dagger_{\vec{p}_n}|0\rangle$. The one-particle states have energy

$$H|\vec{p}\rangle = \frac{\vec{p}^2}{2m}|\vec{p}\rangle \tag{113}$$

which is the non-relativistic dispersion relation. We conclude that quantizing the first order Lagrangian in Equation 106 gives rise to non-relativistic particles of mass m.

A few comments:

- We have a complex field but only a single type of particle. The anti-particle is not in the spectrum. The existence of anti-particles is a consequence of relativity.
- A related fact is that the conserved charge: $Q = \int d^3x : \psi^\dagger\psi :$ is the particle number. This remains conserved even if we include interactions in the Lagrangian of the form

$$\Delta\mathcal{L} = V(\psi^\star\psi) \tag{114}$$

- There is no non-relativistic limit of a real scalar field. In the relativistic theory, the particles are their own anti-particles, and there can be no way to construct a multi-particle theory that conserves particle number.

8.1. Recovering Quantum Mechanics

In quantum mechanics, we talk about the position and momentum operators $\vec{X}$ and $\vec{P}$. In quantum field theory, position is relegated to a label. How do we get back to quantum mechanics? We already have the operator for the total momentum of the field (see Equation 45):

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p} a^\dagger_{\vec{p}} a_{\vec{p}} \tag{115}$$

which, on one-particle states, gives $\vec{P}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$.

It is also easy to construct the position operator. Let us consider the non-relativistic limit. Then the operator

$$\psi^\dagger(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \tag{116}$$

creates a particle with a δ-function localized at $\vec{x}$. We thus write $|\vec{x}\rangle = \psi^\dagger(\vec{x})|0\rangle$. A natural position operator is then

$$\vec{X} = \int d^3x \vec{x} \psi^\dagger(\vec{x})\psi(\vec{x}), \tag{117}$$

so that $\vec{X}|\vec{x}\rangle = \vec{x}|\vec{x}\rangle$.

Let us now construct a state $|\varphi\rangle$ by taking superpositions of one-particle states $|\vec{x}\rangle$,

$$|\varphi\rangle = \int d^3x \varphi(\vec{x})|\vec{x}\rangle. \tag{118}$$

The function $\varphi(\vec{x})$ is what we would usually call the Schrödinger wavefunction (in the position representation). Let's make sure that it indeed satisfies all the right properties. Firstly, it's clear that acting with the position operator $\vec{X}$ has the right action of $\varphi(\vec{x})$,

$$X^i|\varphi\rangle = \int d^3x x^i \varphi(\vec{x})|\vec{x}\rangle. \tag{119}$$

What abut the momentum operator, $\vec{P}$? We will now show that indeed

$$P^i|\varphi\rangle = \int d^3x \left(-i\frac{\partial\varphi}{\partial x^i}\right)|\vec{x}\rangle, \tag{120}$$

which tells us that P^i acts as the familiar derivative on wavefunctions $|\varphi\rangle$. To see this, we write

$$\begin{aligned} P^i|\varphi\rangle &= \int \tfrac{d^3x d^3p}{(2\pi)^3} p^i a^\dagger_{\vec{p}} a_{\vec{p}} \varphi(\vec{x})\psi^\dagger(\vec{x})|0\rangle \\ &= \int \tfrac{d^3x d^3p}{(2\pi)^3} p^i a^\dagger_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \varphi(\vec{x})|0\rangle, \end{aligned} \tag{121}$$

where we have used the relationship $\left[a_{\vec{p}}, \psi^\dagger(\vec{x})\right] = e^{-i\vec{p}\cdot\vec{x}}$, which can be easily checked (using equations 116 and 20). Proceeding with the calculation, we have

$$\begin{aligned} P^i|\varphi\rangle &= \int \tfrac{d^3x d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} \left(i\tfrac{\partial}{\partial x^i} e^{-i\vec{p}\cdot\vec{x}}\right) \varphi(\vec{x})|0\rangle \\ &= \int \tfrac{d^3x d^3p}{(2\pi)^3} e^{-i\vec{p}\cdot\vec{x}} \left(-i\tfrac{\partial\varphi}{\partial x^i}\right) a^\dagger_{\vec{p}}|0\rangle \\ &= \int d^3x \left(-i\tfrac{\partial\varphi}{\partial x^i}\right)|\vec{x}\rangle, \end{aligned} \tag{122}$$

where we have used Equation 116. This result thus confirms the claim in Equation 120. We can thus conclude that when acting on one-particle states, the operators $\vec{X}$ and $\vec{P}$ act as position and momentum operators in quantum mechanics, with $[X^i, P^j]\,|\varphi\rangle = i\delta^{ij}|\varphi\rangle$.

What about the dynamics? How does the wavefunction $\varphi(\vec{x}, t)$ change in time? We can re-write the Hamiltonian in Equation 109 as

$$H = \int d^3x \frac{1}{2m} \nabla\psi^\star \nabla\psi = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}^2}{2m} a^\dagger_{\vec{p}} a_{\vec{p}}, \tag{123}$$

and find that

$$i\frac{\partial\varphi}{\partial t} = -\frac{1}{2m}\nabla^2\varphi. \tag{124}$$

This is the same equation obeyed by the original field, Equation 105!. Except this time, *it really is the Schrödinger equation*, complete with the usual probabilistic interpretation for the wavefunction φ. Note in particular that the conserved charge arising from the Noether current (Equation 107) is $Q = \int d^3x |\varphi(\vec{x})|^2$, which is the total probability.

Historically, the fact that the equation for the classical field (Equation 105) and the one particle wavefunction (Equation 124) coincide caused some confusion. It was thought that perhaps we are quantizing the wavefunction itself and the resulting name "second quantization" is still sometimes used today to mean quantum field theory. It's important to stress that, despite the name, we are not quantizing anything twice! We simply quantize a classical field once. Nonetheless, in practice it's useful to know that if we treat the one-particle Schrödinger equation as the equation for a quantum field then it will give the correct generalization to multi-particle states.

8.2. Interactions

Often in quantum mechanics, we are interested in particles moving in some fixed background potential $V(\vec{x})$. This can be easily incorporated into field theory by working with a Lagrangian with explicit $\vec{x}$ dependence,

$$\mathcal{L} = i\psi^\star\psi - \frac{1}{2m}\nabla\psi^\star\nabla\psi - V(\vec{x})\psi^\star\psi \tag{125}$$

Note that this Lagrangian doesn't respect translational symmetry and we won't have the associated energy-momentum tensor. While such Lagrangians are useful in condensed matter physics, we rarely (or never) come across them in high-energy physics, where all equations obey translational (and Lorentz) invariance.

One can also consider interactions *between* particles. Obviously these are only important for n particle states with $n \geq 2$. We therefore expect them to arise from additions to the Lagrangian of the form

$$\Delta\mathcal{L} = \psi^\star(\vec{x})\psi^\star(\vec{x})\psi(\vec{x})\psi(\vec{x}) \tag{126}$$

which, in the quantum theory, is an operator which destroys two particles before creating two new ones. Such terms in the Lagrangian will indeed lead to inter-particle forces, both in the non-relativistic and relativistic setting.