Gravitational Radiation

Asaf Pe'er[1]

June 16, 2020

This part of the course is based on Refs. [1], [2] and [3].

1. Introduction: linearized gravity

There are many similarities between gravitation and electromagnetism. It is therefore of no surprise that Einstein's equations, like Maxwell's equations, have radiative solution.

When we derived Einstein's equations, we considered the Newtonian limit as a guideline. In this limit, we argued that the gravitational field be **weak**, that it be **static** (no time derivatives), and that **test particles be moving slowly** (See "Equivalence", section 6). Here we consider a less restrictive situation, in which the field is still weak but it can vary with time, and there are no restrictions on the motion of test particles. This will allow us to discuss phenomena which are absent or ambiguous in the Newtonian theory, specifically as gravitational radiation - where the field varies with time.

We start as usual, by decomposing the metric into the flat Minkowski metric plus a small perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \qquad |h_{\mu\nu}| << 1 \ . \tag{1}$$

We restrict ourselves to coordinates in which $\eta_{\mu\nu}$ takes its canonical form, $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. The assumption that $h_{\mu\nu}$ is small allows us to ignore anything that is higher than first order in this quantity, from which we immediately obtain

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \ , \tag{2}$$

where $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. As before, we can raise and lower indices using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, since the corrections would be of higher order in the perturbation.

We want to find the equation of motion obeyed by the perturbations $h_{\mu\nu}$, which come by examining Einstein's equations to first order. We begin with the Christoffel symbols, which are given by

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \tfrac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}) \\ &= \tfrac{1}{2} \eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) \ . \end{aligned} \tag{3}$$

[1]Department of Physics, Bar Ilan University

Since the connection coefficients are first order quantities, the only contribution to the Riemann tensor comes from the derivatives of the Γ's, not the Γ^2 terms. Lowering an index for convenience, and using $\mu \to \nu, \nu \to \sigma, \rho \to \lambda, \lambda \to \alpha$, we obtain

$$\begin{aligned} R_{\mu\nu\rho\sigma} &= \eta_{\mu\lambda}\partial_\rho \Gamma^\lambda_{\nu\sigma} - \eta_{\mu\lambda}\partial_\sigma \Gamma^\lambda_{\nu\rho} \\ &= \tfrac{1}{2}(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\sigma\partial_\nu h_{\mu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma}) \ . \end{aligned} \tag{4}$$

The Ricci tensor comes from contracting over μ and ρ (using $\nu \to \mu, \sigma \to \nu$), giving

$$R_{\mu\nu} = \frac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}) \ , \tag{5}$$

which is manifestly symmetric in μ and ν. In this expression we have defined the trace of the perturbation as $h = \eta^{\mu\nu}h_{\mu\nu} = h^\mu{}_\mu$, and the D'Alembertian is simply the one from flat space, $\Box = \nabla^\mu\nabla\mu = -\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2$. Contracting again we obtain the Ricci scalar,

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h \ . \tag{6}$$

Putting it all together, the Einstein tensor gets the form:

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}R \\ &= \tfrac{1}{2}(\partial_\sigma\partial_\nu h^\sigma{}_\mu + \partial_\sigma\partial_\mu h^\sigma{}_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \eta_{\mu\nu}\Box h) \ . \end{aligned} \tag{7}$$

The linearized field equation is $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $G_{\mu\nu}$ is given by Equation 7 and $T_{\mu\nu}$ is the energy-momentum tensor, calculated to zeroth order in $h_{\mu\nu}$ (the weak field limit implies that we can neglect higher order terms). Here we focus on the vacuum equations, which as usual are just $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is given by Equation 5.

1.1. Choice of gauge: the harmonic gauge

The linearized field equation, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, where $G_{\mu\nu}$ is given by Equation 7, does not yield a unique solution. Given any solution (namely, $h_{\mu\nu}$), we can always generate another solution, $h_{\mu'\nu'}$ by performing a coordinate transformation. Let us show that first. The most general coordinate transformation that leaves the field weak is of the form

$$x^\mu \to x^{\mu\prime} = x^\mu + \epsilon^\mu(x) \tag{8}$$

where $\partial\epsilon^\mu/\partial x^\nu$ is at most of the same order of magnitude as $h_{\mu\nu}$.

The metric in the new coordinate system is

$$g^{\mu'\nu'} = \frac{\partial x^{\mu\prime}}{\partial x^\lambda}\frac{\partial x^{\nu\prime}}{\partial x^\rho} g^{\lambda\rho}, \tag{9}$$

and using $g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu}$, we can write

$$h^{\mu'\nu'} = h^{\mu\nu} - \frac{\partial \epsilon^\mu}{\partial x^\lambda}\eta^{\lambda\nu} - \frac{\partial \epsilon^\nu}{\partial x^\rho}\eta^{\rho\mu}. \tag{10}$$

Thus, if $h_{\mu\nu}$ is a solution to the linearized field equation, so will be

$$h_{\mu'\nu'} = h_{\mu\nu} - \frac{\partial \epsilon_\mu}{\partial x^\nu} - \frac{\partial \epsilon_\nu}{\partial x^\mu}. \tag{11}$$

Note that all 4 $\epsilon_\mu \equiv \epsilon^\nu \eta_{\mu\nu}$ are small but arbitrary functions of x^μ. The fact that the solution is not unique is called **gauge invariance** of the field equation. The transformation in Equation 11 is known as **gauge transformation**. It is analogue to gauge transformation familiar from E&M.

When facing a system that is invariant under some kind of gauge transformations, the natural approach is to fix a gauge. A commonly used gauge is the **harmonic gauge**,

$$\Box x^\mu = 0\ . \tag{12}$$

Here $\Box = \nabla^\mu \nabla_\mu$ is the covariant D'Alembertian. It is crucial to note that each coordinate x^μ is thought of as a scalar function of spacetime. Any function that satisfies $\Box f = 0$ is known as an "harmonic function".

By writing directly the covariant derivatives, the condition in Equation 12 can be written as

$$\begin{aligned} 0 &= \Box x^\mu \\ &= g^{\rho\sigma}\left(\partial_\rho \partial_\sigma x^\mu - \Gamma^\lambda_{\rho\sigma}\partial_\lambda x^\mu\right) \\ &= g^{\rho\sigma}\left(\partial_\rho \delta^\mu_\sigma - \Gamma^\lambda_{\rho\sigma}\delta^\mu_\lambda\right) \\ &= -g^{\rho\sigma}\Gamma^\mu_{\rho\sigma}\ . \end{aligned} \tag{13}$$

The condition $g^{\rho\sigma}\Gamma^\rho_{\rho\sigma} = 0$ is also known as the **Lorentz gauge** but also as **Einstein gauge**.

In the weak field limit (Equation 3), Equation 13 gets the form

$$\frac{1}{2}\eta^{\mu\nu}\eta^{\lambda\rho}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}) = 0\ , \tag{14}$$

or

$$\partial_\mu h^\mu{}_\lambda - \frac{1}{2}\partial_\lambda h = 0\ . \tag{15}$$

In this gauge, the linearized Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ (where $G_{\mu\nu}$ is given in Equation 7) simplifies,

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -16\pi G T_{\mu\nu}\ , \tag{16}$$

while the vacuum equations $R_{\mu\nu} = 0$ (Equaiton 5) take on the elegant form

$$\Box h_{\mu\nu} = 0 \ , \tag{17}$$

which is simply the conventional relativistic wave equation. In deriving equations 16, 17, the first three terms in Equations 5, 7 cancel by the Lorentz gauge, and the 5^{th} term in Equation 7 simplifies by that gauge.

Together, Equations 17 and 15 determine the evolution of a disturbance in the gravitational field in vacuum in the harmonic gauge.

2. Gravitational wave solutions

We begin by defining the "trace-reversed" perturbation $\bar{h}_{\mu\nu}$ by

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h \ . \tag{18}$$

The name makes sense, since $\bar{h}^{\mu}{}_{\mu} = -h^{\mu}{}_{\mu}$.

In terms of $\bar{h}_{\mu\nu}$ the harmonic gauge condition (Equation 15) becomes

$$\partial_{\mu}\bar{h}^{\mu}{}_{\lambda} = 0 \ . \tag{19}$$

The full field equations (16) are

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \ , \tag{20}$$

from which it follows immediately that the vacuum equations are

$$\Box\bar{h}_{\mu\nu} = 0 \ . \tag{21}$$

A look at the linearized Equation 21 reveals that since the flat-space D'Alembertian has the form $\Box = -\partial_t^2 + \nabla^2$, the field equation is in the form of a wave equation for $\bar{h}_{\mu\nu}$.

As all good physicists know, the thing to do when faced with such an equation is to write down complex-valued solutions, and then take the real part at the end. It is easy to recognize that a particularly useful set of solutions to this wave equation are the plane waves, given by

$$\bar{h}_{\mu\nu} = C_{\mu\nu}e^{ik_{\sigma}x^{\sigma}} \ , \tag{22}$$

where $C_{\mu\nu}$ is a constant, symmetric, $(0,2)$ tensor (that provides the amplitudes of the various components of the wave), and k^{σ} is a constant vector known as the **wave vector**. To check that it is a solution, we plug in:

$$\begin{aligned} 0 &= \Box \bar{h}_{\mu\nu} \\ &= \eta^{\rho\sigma}\partial_\rho\partial_\sigma \bar{h}_{\mu\nu} \\ &= \eta^{\rho\sigma}\partial_\rho(ik_\sigma \bar{h}_{\mu\nu}) \\ &= -\eta^{\rho\sigma}k_\rho k_\sigma \bar{h}_{\mu\nu} \\ &= -k_\sigma k^\sigma \bar{h}_{\mu\nu} \ . \end{aligned} \tag{23}$$

Since (for an interesting solution) not all of the components of $\bar{h}_{\mu\nu}$ will be zero everywhere, we must have

$$k_\sigma k^\sigma = 0 \ . \tag{24}$$

The plane wave in Equation 21 is therefore a solution to the linearized equations if the wavevector is null; this is loosely translated into the statement that gravitational waves propagate at the speed of light. The timelike component of the wave vector is the **frequency** of the wave, and we write $k^\sigma = (\omega, k^1, k^2, k^3)$. (More generally, an observer moving with four-velocity U^μ would observe the wave to have a frequency $\omega = -k_\mu U^\mu$.) Then the condition that the wave vector be null becomes

$$\omega^2 = \delta_{ij}k^i k^j \ . \tag{25}$$

Of course, any (possibly infinite) number of distinct plane waves can be added together and will still solve the linear equation 21. Indeed, any solution can be written as such a superposition.

In order to further specify the wave, we note that there are still plenty degrees of freedom. $C_{\mu\nu}$ is a symmetric, $(0, 2)$ tensor, and as such has 10 free coefficients. The null vector k^σ has additional three coefficients. Much of this freedom is due to coordinate freedom and gauge freedom, which we now set about eliminating.

We begin by imposing the harmonic gauge condition, Equation 19. This implies that

$$\begin{aligned} 0 &= \partial_\mu \bar{h}^{\mu\nu} \\ &= \partial_\mu (C^{\mu\nu} e^{ik_\sigma x^\sigma}) \\ &= iC^{\mu\nu}k_\mu e^{ik_\sigma x^\sigma} \ , \end{aligned} \tag{26}$$

which is only true if

$$k_\mu C^{\mu\nu} = 0 \ . \tag{27}$$

We thus find that the wave vector is orthogonal to $C^{\mu\nu}$. These are four equations, which reduce the number of independent components of $C_{\mu\nu}$ from ten to six.

The number of independent componets of $C^{\mu\nu}$ is in fact even lower. Recall that any coordinate transformation of the form

$$x^\mu \to x^\mu + \epsilon^\mu$$

(see Equation 8) will leave the harmonic coordinate condition (Equation 12)

$$\Box x^\mu = 0$$

satisfied as long as

$$\Box \epsilon^\mu = 0 \ . \tag{28}$$

Equation 28 is itself a wave equation for ϵ^μ; Following a similar procedure to what we did above, will impose additional 4 constraints, leaving the number of independent parameters to 2.

Let us see how this works. We write the 4 additional constraints as

$$h_{0\mu} = 0 \quad , h^\mu{}_\mu = 0, \tag{29}$$

from which we get (using Equations 18, 19, and 22)

$$\begin{aligned} \partial_0 \bar{h}^0{}_\lambda = C^0{}_0 i k_0 e^{ik_\sigma x^\sigma} = 0 \\ \partial_i \bar{h}^i{}_\lambda = C^i{}_\lambda i k_i e^{ik_\sigma x^\sigma} = 0 \end{aligned} \tag{30}$$

We thus get

$$C_{00} = 0, \qquad k^\mu C_{\mu\nu} = 0. \tag{31}$$

The last condition means that the gravitational waves are **transverse** - similar to electromagnetic waves.

We are left with two degrees of freedom - namely there are only 2 independent $C_{\mu\nu}$. The easiest way to write them explicitly is to orient the spatial coordinates so that the direction of propagation of the wave is along one axis (z axis). That is,

$$k^\mu = (\omega, 0, 0, k^3) = (\omega, 0, 0, \omega) \ , \tag{32}$$

where we know that $k^3 = \omega$ because the wave vector is null. In this case, the transverse consition, $k^\mu C_{\mu\nu} = 0$ imply that all the components along the z axis vanish,

$$C_{3\nu} = 0 \ . \tag{33}$$

Furthermore, we know from Equation 29 that all components $C_{0\mu} = 0$, and we are thereofe left with only 4 non-zero components for the matrix $C_{\mu\nu}$: $C_{11}, C_{12}, C_{21}, C_{22}$. But $C_{\mu\nu}$ is symmetric, and must be traceless since $h^\mu{}_\mu = 0$; so the **most general** form of the amplitude matrix $C_{\mu\nu}$ is

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{34}$$

Thus, for a plane wave in this gauge travelling in the x^3 direction, the two components C_{11} and C_{12} (along with the frequency ω) completely characterize the wave.

This choice of coordinates in which the transverse and traceless conditions are represented explicitly is called **transverse-traceless gauge**, or simply TT-gauge for short.

The most general solution to the linearized Einstein equation with definite wave number is therefore

$$h_{\mu\nu}(x) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i\omega(z-t)}. \tag{35}$$

where we renamed $h_+ = C_{11}$, $h_\times = C_{12}$ from reasons which will become clear shortly.

2.1. Observational effect

To get a feeling for the physical effects due to gravitational waves, it is best to consider the motion of test particles in the presence of a wave.

It is certainly insufficient to solve for the trajectory of a single particle, since that would only tell us about the values of the coordinates along the world line. To obtain a coordinate-independent measure of the wave's effects, we consider the relative motion of nearby particles, as described by the geodesic deviation equation (see "Curvature", section 9). If we consider some nearby particles with four-velocities described by a single vector field $U^\mu(x)$ and separation vector S^μ, we have

$$\frac{D^2}{D\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho S^\sigma \ . \tag{36}$$

We would like to compute the right-hand side to first order in $h_{\mu\nu}$. If we take the test particles to be moving slowly then we can express the four-velocity as a unit vector in the time direction plus corrections of order $h_{\mu\nu}$ and higher; but we know that the Riemann tensor is already first order, so the corrections to U^ν may be ignored, and we write

$$U^\nu = (1, 0, 0, 0) \ . \tag{37}$$

Therefore we only need to compute ${R^\mu}_{00\sigma}$, or equivalently $R_{\mu 00\sigma}$. From Equation 4 we have

$$R_{\mu 00\sigma} = \frac{1}{2}(\partial_0\partial_0 h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{00} - \partial_\sigma\partial_0 h_{\mu 0} - \partial_\mu\partial_0 h_{\sigma 0}) \ . \tag{38}$$

But $h_{\mu 0} = 0$ (Equation 29), so

$$R_{\mu 00\sigma} = \frac{1}{2}\partial_0\partial_0 h_{\mu\sigma} \ . \tag{39}$$

For slowly-moving particles we have $\tau = x^0 = t$ to lowest order, so the geodesic deviation equation becomes

$$\frac{\partial^2}{\partial t^2} S^\mu = \frac{1}{2} S^\sigma \frac{\partial^2}{\partial t^2} {h^\mu}_\sigma \ . \tag{40}$$

For a wave travelling in the x^3 direction, the results of Equation 35 therefore imply that only S^1 and S^2 will be affected — **the test particles are only disturbed in directions perpendicular to the wave vector**. This is of course familiar from electromagnetism, where the electric and magnetic fields in a plane wave are perpendicular to the wave vector.

We consider a wave characterized by the two numbers, $h_+ = C_{11}$ and $h_\times = C_{12}$. Let's consider their effects separately. Beginning with the $h_\times = 0$ case, we have

$$\frac{\partial^2}{\partial t^2} S^1 = \frac{1}{2} S^1 \frac{\partial^2}{\partial t^2}(h_+ e^{ik_\sigma x^\sigma}) \tag{41}$$

and

$$\frac{\partial^2}{\partial t^2} S^2 = -\frac{1}{2} S^2 \frac{\partial^2}{\partial t^2}(h_+ e^{ik_\sigma x^\sigma}) \ . \tag{42}$$

These can be immediately solved to yield, to lowest order in S^1, S^2,

$$S^1 = \left(1 + \frac{1}{2} h_+ e^{ik_\sigma x^\sigma}\right) S^1(0) \tag{43}$$

and

$$S^2 = \left(1 - \frac{1}{2} h_+ e^{ik_\sigma x^\sigma}\right) S^2(0) \ . \tag{44}$$

Thus, particles initially separated in the x^1 direction will oscillate back and forth in the x^1 direction, and likewise for those with an initial x^2 separation. That is, if we start with a ring of stationary particles in the x-y plane, as the wave passes they will bounce back and forth in the shape of a "+", as shown in Figure 1.

On the other hand, the equivalent analysis for the case where $h_+ = 0$ but $h_\times \neq 0$ would yield the solution

$$S^1 = S^1(0) + \frac{1}{2} h_\times e^{ik_\sigma x^\sigma} S^2(0) \tag{45}$$

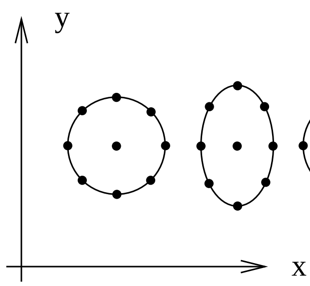

Fig. 1.— The effect of a gravitational wave ith a "+" polarization is to distort a circle of test particles into ellipses oscillating in a "+" pattern.

and

$$S^2 = S^2(0) + \frac{1}{2} h_\times e^{ik_\sigma x^\sigma} S^1(0) \ . \tag{46}$$

In this case the circle of particles would bounce back and forth in the shape of a "×" (see Figure 2)

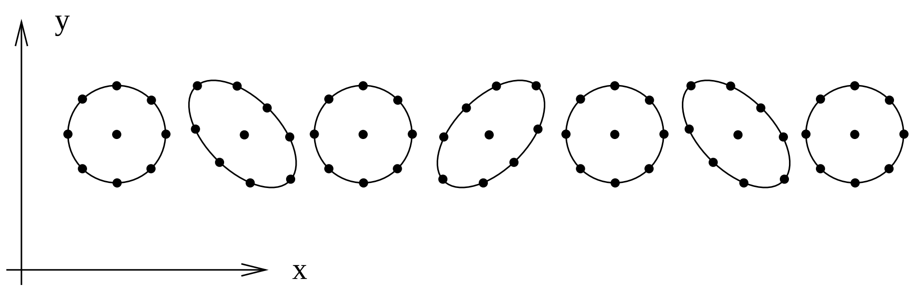

Fig. 2.— The effect of a gravitational wave with × polarization is to distort a circle of test particles into ellipses oscillating in a × pattern.

The notation h_+ and $h_\times$ should therefore be clear now. These two quantities measure the two independent modes of linear polarization of the gravitational wave. One may also consider right- and left-handed circularly polarized modes by defining

$$\begin{aligned} h_R &= \tfrac{1}{\sqrt{2}}(h_+ + ih_\times) \ , \\ h_L &= \tfrac{1}{\sqrt{2}}(h_+ - ih_\times) \ . \end{aligned} \tag{47}$$

The effect of a pure h_R wave is to rotate the particles in a right-handed sense, as shown in Figure 3.

Similarly for the left-handed mode h_L. Note that the individual particles do not travel around the ring; they just move in little epicycles.

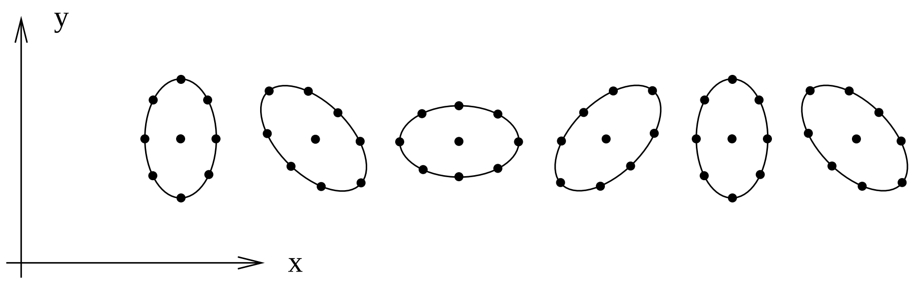

Fig. 3.— The effect of a gravitational wave with R polarization is to distort a circle of test particles into an eliipse that rotates in a right-handed sense.

3. Generation of gravitational waves

We now discuss the question of generating gravitational waves. Here, we have to consider Einstein's equation in matter, $G_{\mu\nu} = 8\pi G T_{\mu\nu}$. Linearizing this equation one gets (see Equation 20)

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \ .$$

The solution to such an equation can be obtained using a Green's function, in precisely the same way as the analogous problem in electromagnetism.

The Green's function $G(x^\sigma - y^\sigma)$ for the D'Alembertian operator $\Box$ is (by definition) the solution of the wave equation in the presence of a delta-function source:

$$\Box_x G(x^\sigma - y^\sigma) = \delta^{(4)}(x^\sigma - y^\sigma) \ , \tag{48}$$

where $\Box_x$ denotes the D'Alembertian with respect to the coordinates x^σ. The usefulness of Green's function resides in the fact that the general solution to an equation such as Equation 20 is given by

$$\bar{h}_{\mu\nu}(x^\sigma) = -16\pi G \int G(x^\sigma - y^\sigma) T_{\mu\nu}(y^\sigma) \ d^4y \ , \tag{49}$$

(proof is immediate; note that no factors of $\sqrt{-g}$ are necessary, since the background is simply flat spacetime.).

The solution to Green's function (Equation 48) can represent "retarded" or "advanced" waves, depending on whether the waves travel forward or backward in time. We are intrested, of course, in retarded Green function, representing waves traveling forward in time - namely, the accumulated effects of signals to the past of the point we are looking at. The solution of Equation 48 is (can be found in mathematical textbook)

$$G(x^\sigma - y^\sigma) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \delta[|\mathbf{x} - \mathbf{y}| - (x^0 - y^0)] \ \theta(x^0 - y^0) \ . \tag{50}$$

Here we have used boldface to denote the spatial vectors $\mathbf{x} = (x^1, x^2, x^3)$ and $\mathbf{y} = (y^1, y^2, y^3)$, with norm $|\mathbf{x} - \mathbf{y}| = [\delta_{ij}(x^i - y^i)(x^j - y^j)]^{1/2}$. The theta function $\theta(x^0 - y^0)$ equals 1 when $x^0 > y^0$, and zero otherwise.

Plugging the Green function (Equation 50) in Equation 49 enable to use the delta-function to perform the integral over y^0, resulting in

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = 4G \int \frac{1}{|\mathbf{x} - \mathbf{y}|} T_{\mu\nu}(t - |\mathbf{x} - \mathbf{y}|, \mathbf{y}) \; d^3y \; , \tag{51}$$

where $t = x^0$. The term "retarded time" is used to refer to the quantity

$$t_r = t - |\mathbf{x} - \mathbf{y}| \; . \tag{52}$$

The interpretation of Equation 51 should be clear: the disturbance in the gravitational field at $(t, \mathbf{x})$ is a sum of the influences from the energy and momentum sources at the point $(t_r, \mathbf{x} - \mathbf{y})$ on the past light cone (see Figure 4).

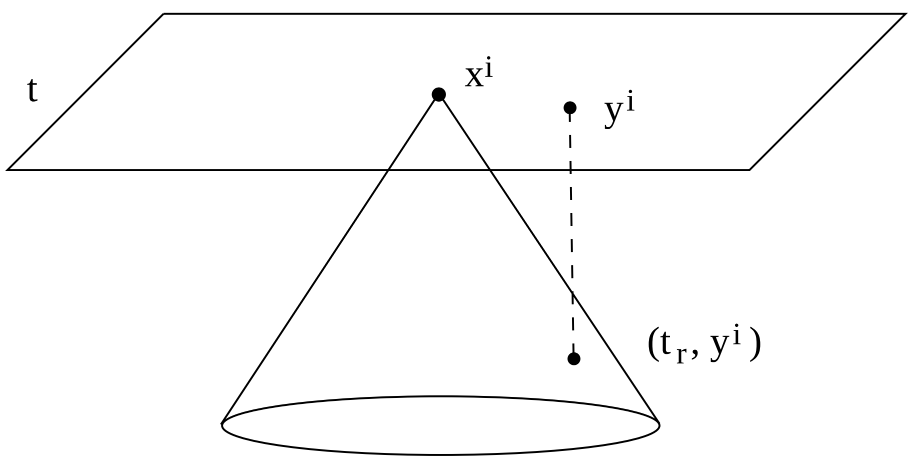

Fig. 4.— Disturbances in the gravitational field at (t, x^i) are calculated in terms of events inside the past light cone.

Let us take this general solution and consider the case where the gravitational radiation is emitted by an isolated source, fairly far away, comprised of nonrelativistic matter. Since we deal with oscillatory phenomenon, it is best to use Fourier transforms.

Given a function of spacetime $\phi(t, \mathbf{x})$, its Fourier transform (and inverse) with respect to time alone are given by

$$\begin{aligned} \widetilde{\phi}(\omega, \mathbf{x}) &= \tfrac{1}{\sqrt{2\pi}} \int dt \; e^{-i\omega t} \phi(t, \mathbf{x}) \; , \\ \phi(t, \mathbf{x}) &= \tfrac{1}{\sqrt{2\pi}} \int d\omega \; e^{i\omega t} \widetilde{\phi}(\omega, \mathbf{x}) \; . \end{aligned} \tag{53}$$

Taking the transform of the metric perturbation, we obtain

$$\begin{aligned}\widetilde{\bar{h}}_{\mu\nu}(\omega,\mathbf{x}) &= \frac{1}{\sqrt{2\pi}}\int dt\; e^{-i\omega t}\bar{h}_{\mu\nu}(t,\mathbf{x}) \\ &= \frac{4G}{\sqrt{2\pi}}\int dt\; d^3y\; e^{-i\omega t}\;\frac{T_{\mu\nu}(t-|\mathbf{x}-\mathbf{y}|,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \\ &= \frac{4G}{\sqrt{2\pi}}\int dt_r\; d^3y\; e^{-i\omega t_r}e^{-i\omega|\mathbf{x}-\mathbf{y}|}\frac{T_{\mu\nu}(t_r,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \\ &= 4G\int d^3y\; e^{-i\omega|\mathbf{x}-\mathbf{y}|}\frac{\widetilde{T}_{\mu\nu}(\omega,\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\;.\end{aligned} \tag{54}$$

Here, we got the second line by using the solution (Equation 51), the third line is a change of variables from t to t_r, and the fourth line is once again the definition of the Fourier transform.

In order to proceed, we assume that the source is isolated, far away, and slowly moving. This means that we can consider the source to be centered at a (spatial) distance R, with the different parts of the source at distances $R+\delta R$ such that $\delta R << R$. Since it is slowly moving, most of the radiation emitted will be at frequencies ω sufficiently low that $\delta R << \omega^{-1}$. (Essentially, light traverses the source much faster than the components of the source itself do.)

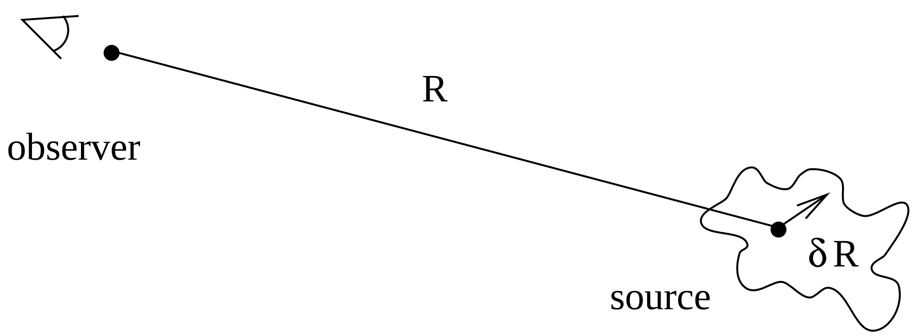

Fig. 5.— A source of size δR, located at distance R from the observer.

Under these approximations, the term $e^{-i\omega|\mathbf{x}-\mathbf{y}|}/|\mathbf{x}-\mathbf{y}|$ can be replaced by $e^{-i\omega R}/R$ and brought outside the integral (see Figure 5). Equation 54 thus become

$$\widetilde{\bar{h}}_{\mu\nu}(\omega,\mathbf{x}) = 4G\frac{e^{-i\omega R}}{R}\int d^3y\;\widetilde{T}_{\mu\nu}(\omega,\mathbf{y})\;. \tag{55}$$

We now have to compute $\widetilde{\bar{h}}_{\mu\nu}(\omega,\mathbf{x})$. However, we can simplify things by writing the harmonic gauge condition, $\partial_\mu\bar{h}^{\mu\nu}(t,\mathbf{x})=0$ in Fourier space:

$$\begin{aligned}0=\partial_\mu\bar{h}^{\mu\nu}(t,\mathbf{x}) &= \frac{1}{\sqrt{2\pi}}\partial_\mu\int d\omega\; e^{i\omega t}\widetilde{\bar{h}}^{\mu\nu}(\omega,x) \\ &= \int d\omega\; e^{i\omega t}\left[(i\omega)\widetilde{\bar{h}}^{0\nu}(\omega,x)+\partial_i\widetilde{\bar{h}}^{i\nu}(\omega,x)\right]\end{aligned}$$

or

$$\widetilde{\bar{h}}^{0\nu} = \frac{i}{\omega}\partial_i\widetilde{\bar{h}}^{i\nu} \ . \qquad (56)$$

Thus, it is sufficient to calculate the spacelike components of $\widetilde{\bar{h}}_{\mu\nu}(\omega, \mathbf{x})$.

From Equation 55 we integrate over the spacelike components of $\widetilde{T}_{\mu\nu}(\omega, \mathbf{y})$. Integrating by parts in reverse, we get

$$\int d^3y \ \widetilde{T}^{ij}(\omega, \mathbf{y}) = \int \partial_k(y^i\widetilde{T}^{kj}) \ d^3y - \int y^i(\partial_k\widetilde{T}^{kj}) \ d^3y \ . \qquad (57)$$

The first term on the right is a surface integral which vanishes for an isolated source. The second can be related to $\widetilde{T}^{0j}$ by the Fourier-space version of $\partial_\mu T^{\mu\nu} = 0$,

$$\partial_k\widetilde{T}^{k\mu} = -i\omega\widetilde{T}^{0\mu} \ . \qquad (58)$$

Thus,

$$\begin{aligned} \int d^3y \ \widetilde{T}^{ij}(\omega, \mathbf{y}) &= i\omega \int y^i\widetilde{T}^{0j} \ d^3y \\ &= \tfrac{i\omega}{2} \int (y^i\widetilde{T}^{0j} + y^j\widetilde{T}^{0i}) \ d^3y \\ &= \tfrac{i\omega}{2} \int \left[\partial_l(y^iy^j\widetilde{T}^{0l}) - y^iy^j(\partial_l\widetilde{T}^{0l})\right] \ d^3y \\ &= -\tfrac{\omega^2}{2} \int y^iy^j\widetilde{T}^{00} \ d^3y \ . \end{aligned} \qquad (59)$$

The second line is justified since we know that the left hand side is symmetric in i and j, while the third and fourth lines are simply repetitions of reverse integration by parts and conservation of $T^{\mu\nu}$.

It is therefore conventional to define the **quadrupole moment tensor** of the energy density of the source,

$$I_{ij}(t) = \int y^iy^jT^{00}(t, \mathbf{y}) \ d^3y \ , \qquad (60)$$

a constant tensor on each surface of constant time. In terms of the Fourier transform of the quadrupole moment, our solution takes on the compact form

$$\widetilde{\bar{h}}_{ij}(\omega, \mathbf{x}) = -2G\omega^2\frac{e^{-i\omega R}}{R}\widetilde{I}_{ij}(\omega) \ , \qquad (61)$$

or, transforming back to t,

$$\begin{aligned} \bar{h}_{ij}(t, \mathbf{x}) &= -\tfrac{1}{\sqrt{2\pi}}\tfrac{2G}{R} \int d\omega \ e^{i\omega(t-R)}\omega^2\widetilde{I}_{ij}(\omega) \\ &= \tfrac{1}{\sqrt{2\pi}}\tfrac{2G}{R}\tfrac{d^2}{dt^2} \int d\omega \ e^{i\omega t_r}\widetilde{I}_{ij}(\omega) \\ &= \tfrac{2G}{R}\tfrac{d^2I_{ij}(t_r)}{dt^2} \ , \end{aligned} \qquad (62)$$

where as before $t_r = t - R$.

We thus obtained the **quadrupole formula**,

$$\bar{h}_{ij}(t,\mathbf{x}) = \frac{2G}{R}\frac{d^2 I_{ij}(t_r)}{dt^2}.$$

The gravitational wave produced by an isolated nonrelativistic object is therefore proportional to the second derivative of the quadrupole moment of the energy density at the point where the past light cone of the observer intersects the source.

This result can be compared to the leading contribution to electromagnetic radiation, that comes from the changing *dipole* moment of the charge density. The difference can be traced back to the universal nature of gravitation. A changing dipole moment corresponds to motion of the center of density — charge density in the case of electromagnetism, energy density in the case of gravitation. While there is nothing to stop the center of charge of an object from oscillating, oscillation of the center of mass of an isolated system violates conservation of momentum. The quadrupole moment, which measures the shape of the system, is generally smaller than the dipole moment, and for this reason (as well as the weak coupling of matter to gravity) gravitational radiation is typically much weaker than electromagnetic radiation.

3.1. Example

One of the most important sources of gravitational radiation is due to the motion of binary stars - two stars in orbit around each other. For simplicity let us consider two stars of mass M in a circular orbit in the x^1-x^2 plane, at distance R from their common center of mass (see Figure 6).

In reality, it is safe to treat the motion of the stars in the Newtonian approximation, namely Keplerian orbits. Circular orbits are most easily characterized by equating the force due to gravity to the outward "centrifugal" force:

$$\frac{GM^2}{(2r)^2} = \frac{Mv^2}{r} \ ,$$

which gives the velocity

$$v = \left(\frac{GM}{4r}\right)^{1/2} \ . \tag{63}$$

The time it takes to complete a single orbit is therefore

$$T = \frac{2\pi r}{v} \ . \tag{64}$$

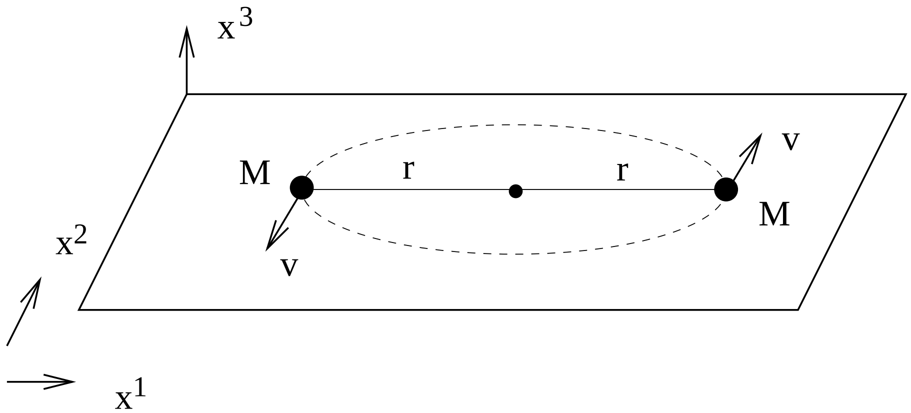

Fig. 6.— A binary star system. Two stars of equal masses M orbit in the $x^1 - x^2$ plane, with an orbital radius R.

The useful quantity is the angular frequency of the orbit,

$$\Omega = \frac{2\pi}{T} = \left(\frac{GM}{4r^3}\right)^{1/2} . \tag{65}$$

In terms of Ω we can write down the explicit path of the stars: star a,

$$x_a^1 = r\cos\Omega t \ , \qquad x_a^2 = r\sin\Omega t \ , \tag{66}$$

and star b,

$$x_b^1 = -r\cos\Omega t \ , \qquad x_b^2 = -r\sin\Omega t \ . \tag{67}$$

The corresponding energy density is

$$T^{00}(t,\mathbf{x}) = M\delta(x^3)\left[\delta(x^1 - r\cos\Omega t)\delta(x^2 - r\sin\Omega t) + \delta(x^1 + r\cos\Omega t)\delta(x^2 + r\sin\Omega t)\right] . \tag{68}$$

We use the δ-functions to integrate this straightforwardly, and obtain the quadrupole moment tensor from Equation 60:

$$\begin{aligned} I_{11} &= 2Mr^2\cos^2\Omega t = Mr^2(1+\cos 2\Omega t), \\ I_{22} &= 2Mr^2\sin^2\Omega t = Mr^2(1-\cos 2\Omega t), \\ I_{12} = I_{21} &= 2Mr^2(\cos\Omega t)(\sin\Omega t) = Mr^2\sin 2\Omega t, \\ I_{i3} &= 0 \ . \end{aligned} \tag{69}$$

Using this result in the quadrupole formula (Equation 62) gives the components of the metric perturbation,

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{8GM}{R}\Omega^2 r^2 \begin{pmatrix} -\cos 2\Omega t_r & -\sin 2\Omega t_r & 0 \\ -\sin 2\Omega t_r & \cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{70}$$

The remaining components of $\bar{h}_{\mu\nu}$ could be derived from demanding that the harmonic gauge condition be satisfied.

4. Energy loss due to gravitational radiation

Emission of gravitational waves is accompanied by energy losses. However, we need to be careful in defining how to measure the true energy of a gravitational field. In the weak field limit, this implies associating an energy-momentum tensor to the fluctuations $h_{\mu\nu}$, in an analogue way to electromagnetic field.

At the technical level, we have to be careful. The stress-energy tensor for the electromagnetic field is **quadratic** in the field. However, in the weak field limit, we only kept terms that are linear in the metric perturbation. Thus, assuming that the stress-energy tensor associated with gravitational waves is also quadratic in the field (as it is), we have to extend the calculations to (at least) second order term in $h_{\mu\nu}$.

With this in mind, consider Einstein's equation (in vacuum), $R_{\mu\nu} = 0$ to second order, and see how the results csan be interpreted in terms of an energy-momentum tensor for the gravitational field. We expand both the metric and the Ricci tensor,

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu}, \\ R_{\mu\nu} &= R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(02}_{\mu\nu} \end{aligned} \tag{71}$$

where $R^{(1)}_{\mu\nu}$ is taken to be of the same order as $h^{(1)}_{\mu\nu}$, while $R^{(2)}_{\mu\nu}$ and $h^{(2)}_{\mu\nu}$ are of the order $\left(h^{(1)}_{\mu\nu}\right)^2$.

The zeroth order equation, $R^{(0)}_{\mu\nu} = 0$ is automatically satisfied since we work in a flat background. The dirst order equation,

$$R^{(1)}_{\mu\nu}[h^{(1)}] = 0$$

determines the first order perturbation $h^{(1)}_{\mu\nu}$ up to (unavoidable) gauge transformation. The second order perturbation, $h^{(2)}_{\mu\nu}$ is determined by the second order equation:

$$R^{(1)}_{\mu\nu}[h^{(2)}] + R^{(2)}_{\mu\nu}[h^{(1)}] = 0 \tag{72}$$

The notation $R^{(1)}_{\mu\nu}[h^{(2)}]$ refer to the part of the expanded Ricci tensor that is linear in the metric perturbation, applied to a seocnd order perturbation; similarly, $R^{(2)}_{\mu\nu}[h^{(1)}]$ is the quadratic part of the Ricci tensor applied to first order perturbation, $h^{(1)}_{\mu\nu}$. There are no cross terms, as those will be of higher order. Written explicitely, the term $R^{(2)}_{\mu\nu}$ takes the form

$$\begin{aligned} R^{(2)}_{\mu\nu} \;=\; & \tfrac{1}{2}h^{\rho\sigma}\partial_\mu\partial_\nu h_{\rho\sigma} - h^{\rho\sigma}\partial_\rho\partial_{(\mu}h_{\nu)\sigma} + \tfrac{1}{4}(\partial_\mu h_{\rho\sigma})\partial_\nu h^{\rho\sigma} + (\partial^\sigma h^\rho{}_\nu)\partial_{[\sigma}h_{\rho]\mu} \\ & + \tfrac{1}{2}\partial_\sigma(h^{\rho\sigma}\partial_\rho h_{\mu\nu}) - \tfrac{1}{4}(\partial_\rho h_{\mu\nu})\partial^\rho h - (\partial_\sigma h^{\rho\sigma} - \tfrac{1}{2}\partial^\rho h)\partial_{(\mu}h_{\nu)\rho} \ . \end{aligned}$$

where $\partial_{(\mu}h_{\nu)\rho} = \frac{1}{2}\left(\partial_\mu h_{\nu\rho} + \partial_\nu h_{\mu\rho}\right)$ is the symmetric part of the tensor.

The most straight forward way is to write Einstein's equation in vacuum as $G_{\mu\nu} = 0$, rather than $R_{\mu\nu} = 0$ - the equations are equivalent, but the first way is more insightful. We get, to second order,

$$R^{(1)}_{\mu\nu}[h^{(2)}] - \frac{1}{2}\eta^{\rho\sigma}R^{(1)}_{\rho\sigma}[h^{(2)}]\eta_{\mu\nu} = 8\pi G\tilde{T}_{\mu\nu}, \tag{73}$$

where we have defined

$$\tilde{T}_{\mu\nu} \equiv -\frac{1}{8\pi G}\left\{R^{(2)}_{\mu\nu}[h^{(1)}] - \frac{1}{2}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}[h^{(1)}]\eta_{\mu\nu}\right\} \tag{74}$$

This notation is of course meant to suggest that we think of $\tilde{T}_{\mu\nu}$ as an energy-momentum tensor associated with the gravitational field (at least in the weak field limit regime). The Bianchi identity $\partial_\mu G^{\mu\nu} = 0$ further implies that $\tilde{T}_{\mu\nu}$ is conserved in the flat space, namely

$$\partial_\mu \tilde{T}^{\mu\nu} = 0 \ . \tag{75}$$

Clearly, there are limitations to the interpretation of $\tilde{T}_{\mu\nu}$ as an energy-momentum tensor. It is not a tensor at all in the full theory; furthermore, it is not invariant under gauge transformations. Furtunately, it is possible to construct global quantities which are invariant under certain special kinds of gauge transformations. These include the total energy on a surface Σ of constant time,

$$E = \int_\Sigma \tilde{T}_{00}\ d^3x \ , \tag{76}$$

and the total energy radiated through to infinity,

$$\Delta E = \int_S P\ dt = \int dt \int_{S_\infty} \tilde{T}_{0\mu}n^\mu\ r^2\ d\Omega \ . \tag{77}$$

Here, the integral is taken over the two sphere S_∞, and n^μ is a unit space-like vector normal to S_∞, whose components in polar coordinates $[t, r, \theta, \phi]$ are simply

$$n^\mu = (0, 1, 0, 0).$$

Evaluating these formulas in terms of the quadrupole moment of a radiating source involves a lengthy and very technical calculation, which I skip; it is provided in Caroll's book (chapter 7). For the rotating binary star example given in section 3.1, the total radiated power is

$$P = -\frac{128}{5} G M^2 R^4 \Omega^6 \tag{78}$$

or, using equation 65 for the frequency,

$$P = \frac{2}{5} \frac{G^4 M^5}{R^5} \tag{79}$$

This energy loss through gravitational waves has actually been observed. In 1974 Hulse and Taylor discovered a binary system, PSR1913+16, in which both stars are very small (so classical effects are negligible, or at least under control) and one is a pulsar. The period of the orbit is eight hours, extremely small by astrophysical standards. The fact that one of the stars is a pulsar provides a very accurate clock, with respect to which the change in the period as the system loses energy can be measured. The result is consistent with the prediction of general relativity for energy loss through gravitational radiation. Hulse and Taylor were awarded the Nobel Prize in 1993 for this discovery.

REFERENCES

[1] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons), chapter 10.

[2] S. Carroll, *Lecture Notes on General Relativity*, part 6. Weak Fieldsand Gravitational Radiation
(http://preposterousuniverse.com/grnotes/).

[3] J. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Addison-Wesley), chapter 21.