Interacting Fields

Asaf Pe'er[1]

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This part of the course is based on Refs. [1] and [2].

1. Introduction

The free field theories that we have discussed so far are very special: we can determine their spectrum, but nothing interesting then happens. They have particle excitations, but these particles don't interact with each other.

Generally, free field theories are characterized by a quadratic Lagrangian, which result in a linear equation of motion; these equations then have exact solutions (in fact we know how to solve exactly only 2 systems: the Hydrogen atom and the harmonic oscillator). Here, we will consider theories which are **very close** to free theories - which we can fully solve - and add a small perturbation.

Our theories will be more complicated, and include interaction terms. These will take the form of higher order terms in the Lagrangian.

We start by asking: "what kind of small perturbations we can add to the theory". For example, consider the Lagrangian for a real scalar field,

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 - \sum_{n\geq 3} \frac{\lambda_n}{n!}\phi^n. \qquad (1)$$

The coefficients λ_n are called **coupling constants** (the factor $n!$ is a mere convention). We can now ask: what restrictions do we have on λ_n to ensure that the additional terms are *small perturbations?*

You might think that we need simply make "$\lambda_n \ll 1$". But this is true only for **dimensionless parameter**, and this is not the case. To see this, let's do some dimensional analysis. Firstly, note that the action has dimensions of angular momentum ($[S] = L^2MT^{-1}$), or, equivalently, the same dimensions as $\hbar$. Since we have set $\hbar = 1$, using the convention described in the introduction, we have $[S] = 0$. With $S = \int d^4x\mathcal{L}$, and $[d^4x] = -4$, the

[1]Physics Dep., University College Cork

Lagrangian density must therefore have dimension

$$[\mathcal{L}] = 4. \tag{2}$$

Now let us look at the Lagrangian in Equation 1. since $[\partial_\mu] = 1$, we can read off the mass dimensions of all the factors to find

$$[\phi] = 1 \quad , \quad [m] = 1 \quad , \quad [\lambda_n] = 4 - n \tag{3}$$

So now we see why we can't simply say we need $\lambda_n \ll 1$: this statement only makes sense for dimensionless quantities !.

So, what does "small λ_n" means? The various terms, parameterized by λ_n, fall into three different categories:

- $[\lambda_3] = 1$: For this term, the dimensionless parameter is λ_3/E, where E has dimensions of mass. Typically in quantum field theory, E **is the energy scale of the process of interest**: that is, E is the energy of the field that solves the un-perturbed equations of motion.

  This means that $\lambda_3\phi^3/3!$ is a small perturbation at high energies, $E \gg \lambda_3$, but a large perturbation at low energies $E \ll \lambda_3$. Terms that we add to the Lagrangian with this behavior are called **relevant** because they are most relevant at low energies (which, after all, is where most of the physics we see lies).

  Important point: Note that in a relativistic theory, $E > m$, so we can always make this perturbation small by taking $\lambda_3 \ll m$.

- $[\lambda_4] = 0$: This term is small if $\lambda_4 \ll 1$. Such perturbations are called **marginal**.

- $[\lambda_n] < 0$ for $n \geq 5$: The dimensionless parameter is $(\lambda_n E^{n-4})$, which is small at low-energies and large at high energies. Such perturbations are thus called **irrelevant**.

As we will see later, it is typically *impossible to avoid high energy processes* in quantum field theory. (We have already seen a glimpse of this in computing the vacuum energy). This means that we might expect problems with irrelevant operators. Indeed, these lead to **"non-renormalizable"** field theories in which one cannot make sense of the infinities at arbitrarily high energies. This doesn't necessarily mean that the theory is useless; just that it is incomplete at some energy scale.

Let me note however that the naive assignment of relevant, marginal and irrelevant is not always fixed in stone: quantum corrections can sometimes change the character of an operator.

1.1. Aside: why Quantum Field Theory is Simple

Typically in a quantum field theory, only the relevant and marginal couplings are important. This is basically because, as we have seen above, the irrelevant couplings become small at low-energies. This is a huge help: of the infinite number of interaction terms that we could write down, only a handful are actually needed (just two in the case of the real scalar field described above).

Let's look at this a little more. Suppose that we some day discover the true superduper "theory of everything unimportant" that describes the world at very high energy scales, say the GUT scale, or the Planck scale. Whatever this scale is, let's call it Λ. It is an energy scale, so $[\Lambda] = 1$. Now we want to understand the laws of physics down at our puny energy scale $E \ll \Lambda$. Let's further suppose that down at the energy scale E, the laws of physics are described by a real scalar field. (They are not of course: they are described by non-Abelian gauge fields and fermions, but the same argument applies in that case as well). This scalar field will have some complicated interaction terms in the form of Equation 1, where the precise form is dictated by all the stuff that is going on in the high energy superduper theory. What are these interactions? Well, we could write our *dimensionful* coupling constants λ_n in terms of *dimensionless* couplings g_n, multiplied by a suitable power of the relevant scale Λ,

$$\lambda_n = \frac{g_n}{\Lambda^{n-4}} \tag{4}$$

The exact values of dimensionless couplings g_n depend on the details of the high-energy superduper theory, but typically one expects them to be of order 1: $g_n \sim \mathcal{O}(1)$. This means that for experiments at small energies $E \ll \Lambda$, the interaction terms of the form ϕ^n with $n > 4$ will be suppressed by powers of $(E/\Lambda)^{n-4}$. This is usually a suppression by many orders of magnitude. (e.g for the energies E explored at the LHC, $E/M_{pl} \sim 10^{-16}$). It is this simple argument, based on dimensional analysis, that ensures that we need only focus on the first few terms in the interaction: those which are relevant and marginal. It also means that if we only have access to low-energy experiments (which we do!), it's going to be very difficult to figure out the high energy theory (which it is!), because its effects are highly diluted except for the relevant and marginal interactions. The discussion given above is a poor man's version of the ideas of *effective field theory* and *Wilson's renormalization group.*

1.2. Examples of Weakly Coupled Theories

In this course we will study only **weakly coupled field theories** i.e. ones that can truly be considered as small perturbations of the free field theory at all energies. In this section, we will look at two types of interactions:

(1): ϕ^4 Theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4, \tag{5}$$

with $\lambda \ll 1$.

We can get a hint for what the effects of this extra term will be. Expanding out ϕ^4 in terms of a and $a^\dagger$, we see a sum of interactions that look like

$$a^\dagger_{\vec{p}}a^\dagger_{\vec{p}}a^\dagger_{\vec{p}}a^\dagger_{\vec{p}} \quad \text{and} \quad a^\dagger_{\vec{p}}a^\dagger_{\vec{p}}a^\dagger_{\vec{p}}a_{\vec{p}} \quad \text{etc.} \tag{6}$$

These will create and destroy particles. This suggests that the ϕ^4 Lagrangian describes a theory in which particle number is not conserved. Indeed, we could check that the number operator N now satisfies $[H, N] \neq 0$.

(2) Scalar Yukawa Theory:

$$\mathcal{L} = \partial_\mu\psi^\star\partial^\mu\psi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - M^2\psi^\star\psi - \frac{1}{2}m^2\phi^2 - g\psi^\star\psi\phi, \tag{7}$$

with $g \ll M, m$.

This theory couples a complex scalar ψ to a real scalar ϕ. While the individual particle numbers of ψ and ϕ are no longer conserved, we do still have a symmetry rotating the phase of ψ, ensuring the existence of the charge Q defined in "Free fields" (equation 75), such that $[Q, H] = 0$. This means that the number of ψ particles minus the number of ψ anti-particles is conserved. It is common practice to denote the anti-particle as $\bar{\psi}$.

The scalar Yukawa theory has a slightly worrying aspect: the potential has a stable local minimum at $\phi = \psi = 0$, but is unbounded below for large enough $-g\phi$. This means we shouldnt try to push this theory too far.

A comment on Strongly Coupled Field Theories

In this course we restrict attention to weakly coupled field theories where we can use perturbative techniques. The study of strongly coupled field theories is much more difficult, and one of the major research areas in theoretical physics. For example, some of the amazing things that can happen include:

- **Charge Fractionalization:** Although electrons have electric charge 1, under the right conditions the elementary excitations in a solid have fractional charge $1/N$ (where $N \in 2\mathbf{Z} + 1$). For example, this occurs in the fractional quantum Hall effect.

- **Confinement:** The elementary excitations of quantum chromodynamics (QCD) are **quarks**. But they **never** appear on their own, only in groups of three (in a baryon) or with an anti-quark (in a meson). They are confined.
- **Emergent Space:** There are field theories in four dimensions which at strong coupling become quantum gravity theories in ten dimensions! The strong coupling effects cause the excitations to act as if they are gravitons moving in higher dimensions. This is quite extraordinary and still poorly understood. Its called the AdS/CFT correspondence.

2. The Interaction Picture.

There is a useful trick to deal with small perturbations in quantum mechanics, namely to describe situations where we have small perturbations to a well-understood Hamiltonian. This is known as **the interaction picture**.

Let's return to the familiar ground of quantum mechanics with a finite number of degrees of freedom for a moment. In the Schrödinger picture, the states evolve as

$$i\frac{d|\psi\rangle_S}{dt} = H|\psi\rangle_S, \tag{8}$$

while the operators $\mathcal{O}_S$ are **independent of time**.

In contrast, in the Heisenberg picture the states are fixed and the operators change in time,

$$\begin{aligned} \mathcal{O}_H &= e^{iHt}\mathcal{O}_S e^{-iHt}, \\ |\psi\rangle_H &= e^{iHt}|\psi\rangle_S. \end{aligned} \tag{9}$$

The **interaction picture** is a hybrid of the two. We split the Hamiltonian up as

$$H = H_0 + H_{\text{int}}, \tag{10}$$

where H_0 is the Hamiltonian of the free field, which we can thus fully solve for, while H_{int} is a small perturbation to it.

The time dependence of *operators* is governed by H_0, while the time dependence of *states* is governed by H_{int}. Although the split into H_0 and H_{int} is arbitrary, it's useful when H_0 is soluble (for example, when H_0 is the Hamiltonian for a free field theory), while H_{int} takes all the more-than quadratic terms.

The states and operators in the interaction picture will be denoted by a subscript I and are given by,

$$\begin{aligned} |\psi(t)\rangle_I &= e^{iH_0t}|\psi(t)\rangle_S, \\ \mathcal{O}_I(t) &= e^{iH_0t}\mathcal{O}_S e^{-iH_0t}. \end{aligned} \tag{11}$$

This last equation also applies to $H_{\rm int}$, which is time dependent.

The interaction part of the Hamiltonian in the interaction picture is

$$H_I \equiv (H_{\rm int})_I = e^{iH_0t}(H_{\rm int})_S e^{-iH_0t}. \tag{12}$$

The Schrödinger equation for states in the interaction picture can be derived starting from the Schrödinger picture:

$$\begin{aligned} i\tfrac{d|\psi\rangle_S}{dt} = H_S|\psi\rangle_S \quad &\Rightarrow \quad i\tfrac{d}{dt}\left(e^{-iH_0t}|\psi(t)\rangle_I\right) = (H_0 + H_{\rm int})_S \left(e^{-iH_0t}|\psi(t)\rangle_I\right) \\ &\Rightarrow \quad i\tfrac{d|\psi\rangle_I}{dt} = e^{iH_0t}(H_{\rm int})_S e^{-iH_0t}|\psi\rangle_I. \end{aligned} \tag{13}$$

Combined with Equation 12 we thus find that

$$i\frac{d|\psi\rangle_I}{dt} = H_I(t)|\psi\rangle_I. \tag{14}$$

2.1. Dyson's Formula

We are looking for a solution to equation 14. Let us write a solution as

$$|\psi(t)\rangle_I = U(t,t_0)|\psi(t_0)\rangle_I, \tag{15}$$

where $U(t,t_0)$ is a unitary **time evolution operator**, such that $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$, and $U(t,t) = 1$. Then the interaction picture Schrödinger equation (equation 14) require that

$$i\frac{dU}{dt} = H_I(t)U. \tag{16}$$

How do we solve Equation 16, given that both $H_I(t)$ and U are **operators**? If H_I were a function, then we could simply solve this:

$$U(t,t_0) \stackrel{?}{=} \exp\left(-{\rm i}\int_{\rm t_0}^{\rm t} {\rm H_I(t')dt'}\right). \tag{17}$$

However, life is not so easy. Our Hamiltonian H_I is an operator, and we have ordering issues.

The main problem is that the operators don't commute. Particularly, $H_I(t_1)$ doesn't commute with $H_I(t_2 \neq t_1)$. Let us see why this is a problem. The exponential of an operator is defined in terms of the Taylor expansion:

$$\exp\left(-{\rm i}\int_{\rm t_0}^{\rm t} {\rm H_I(t')dt'}\right) = 1 - {\rm i}\int_{\rm t_0}^{\rm t} {\rm H_I(t')dt'} + \frac{(-{\rm i})^2}{2}\left(\int_{\rm t_0}^{\rm t} {\rm H_I(t')dt'}\right)^2 + \ldots \tag{18}$$

But when we try to differentiate this with respect to t, we find that the quadratic term gives us

$$-\frac{1}{2}\left(\int_{t_0}^{t} H_I(t')dt'\right) H_I(t) - \frac{1}{2}H_I(t)\left(\int_{t_0}^{t} H_I(t')dt'\right). \tag{19}$$

The second term looks good, since it will give part of the $H_I(t)U$ that we need on the right-hand side of Equation 16. However, the first term is no good, since $H_I(t)$ sits in the wrong side of the integral term, and we cannot commute it through since $[H_I(t'), H_I(t)] \neq 0$ when $t' \neq t$. We thus need to find a way around this.

Claim: The solution to Equation 16 is given by **Dyson's Formula**,

$$U(t,t_0) = T\exp\left(-\mathrm{i}\int_{\mathrm{t_0}}^{\mathrm{t}} \mathrm{H_I(t')dt'}\right), \tag{20}$$

where T stands for **time ordering**, where operators evaluated at later times are placed to the left:

$$T\left[\mathcal{O}_1(t_1)\mathcal{O}_2(t_2)\right] = \begin{cases} \mathcal{O}_1(t_1)\mathcal{O}_2(t_2) & t_1 > t_2 \\ \mathcal{O}_2(t_2)\mathcal{O}_1(t_1) & t_2 > t_1. \end{cases} \tag{21}$$

We can now expand the expression in Equation 20, to get

$$\begin{aligned} U(t,t_0) &= 1 - i\int_{t_0}^{t} H_I(t')dt' + \frac{(-i)^2}{2}T\left[\int_{t_0}^{t} dt' H_I(t')\int_{t_0}^{t} dt'' H_I(t'')\right] + ... \\ &= 1 - i\int_{t_0}^{t} H_I(t')dt' + \frac{(-i)^2}{2}T\left[\int_{t_0}^{t} dt' H_I(t')\left(\int_{t_0}^{t'} dt'' H_I(t'') + \int_{t'}^{t} dt'' H_I(t'')\right)\right] + ... \\ &= 1 - i\int_{t_0}^{t} H_I(t')dt' + \frac{(-i)^2}{2}\left[\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt'' H_I(t')H_I(t'') + \int_{t_0}^{t} dt'\int_{t'}^{t} dt'' H_I(t'')H_I(t')\right] + ... \end{aligned}$$

In fact, the last two terms double up since

$$\begin{aligned} \int_{t_0}^{t} dt'\int_{t'}^{t} dt'' H_I(t'')H_I(t') &= \int_{t_0}^{t} dt''\int_{t_0}^{t''} dt' H_I(t'')H_I(t') \\ &= \int_{t_0}^{t} dt'\int_{t_0}^{t'} dt'' H_I(t')H_I(t'') \end{aligned} \tag{22}$$

where the range of integration in the first expression is over $t'' \geq t'$, while in the second expression it is $t' \leq t''$ which is, of course, the same thing. The final expression is the same as the second expression by a simple relabeling. This means that we can write

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t')dt' + (-i)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt'' H_I(t')H_I(t'') + ... \tag{23}$$

Proof. The proof of Dyson's formula is in fact very simple. First, observe that **under the T sign, all operators commute** (since their order is already fixed by the T sign). Thus,

$$\begin{aligned} i\frac{\partial}{\partial t}T\exp\left(-\mathrm{i}\int_{\mathrm{t_0}}^{\mathrm{t}} \mathrm{dt'H_I(t')}\right) &= T\left[H_I(t)\exp\left(-\mathrm{i}\int_{\mathrm{t_0}}^{\mathrm{t}} \mathrm{dt'H_I(t')}\right)\right] \\ &= H_I(t)T\exp\left(-\mathrm{i}\int_{\mathrm{t_0}}^{\mathrm{t}} \mathrm{dt'H_I(t')}\right) \end{aligned} \tag{24}$$

since t, being the upper limit of the integral, is the latest time so $H_I(t)$ can be pulled out to the left. This completed the proof.

Before moving on, I should confess that Dysons formula is rather formal. It is typically very hard to compute time ordered exponentials in practice. The power of the formula comes from the expansion which is valid when H_I is small and is very easily computed.

3. A First Look at Scattering

Let us now apply the interaction picture to field theory, starting with the interaction Hamiltonian for our scalar Yukawa theory (whose Lagrangian we saw in Equation 7),

$$H_{\text{int}} = g \int d^3x \psi^\dagger \psi \phi. \tag{25}$$

The interaction term is **relevant**: g has a dimension of mass. We will thus take $g \ll M, m$, to guarantee a weak coupling between the scalar and the complex fields.

Unlike the free theories discussed earlier, this interaction **doesn't conserve particle number**, allowing particles of one type to morph into others. To see why this is, we use the interaction picture and follow the evolution of the state: $|\psi(t)\rangle = U(t, t_0)|\psi(t_0)\rangle$, where $U(t, t_0)$ is given by Dyson's formula, Equation 20, which is an expansion in powers of H_{int}. However, H_{int} contains creation and annihilation operators for each type of particle. In particular the terms:

- $\phi \sim a + a^\dagger$: This operator can create or destroy ϕ particles. Let's call them **mesons**.
- $\psi \sim b + c^\dagger$: This operator can destroy ψ particles through b, and create anti-particles through $c^\dagger$. Let's call these particles **nucleons**. Of course, in reality nucleons are spin 1/2 particles, and don't arise from the quantization of a scalar field. But we will treat our scalar Yukawa theory as a toy model for nucleons interacting with mesons.
- $\psi^\dagger \sim b^\dagger + c$: This operator can create nucleons through $b^\dagger$, and destroy anti-nucleons through c.

Importantly, $Q = N_c - N_b$ remains conserved in the presence of H_{int}. At first order in perturbation theory (the first term in the Taylor series of the exponent), we find terms that are proportional to H_{int}, namely terms like $\psi^\dagger \psi \phi \sim c^\dagger b^\dagger a$. This kills a meson, producing a nucleon-anti-nucleon pair. It will contribute to meson decay $\phi \to \psi\bar{\psi}$.

At second order in perturbation theory, we will have more complicated terms that arise from $(H_{\rm int})^2$, for example $(c^\dagger b^\dagger a)(cba^\dagger)$. This term will give contributions to scattering processes $\psi\bar{\psi} \to \phi \to \psi\bar{\psi}$.

The rest of this section is devoted to computing what we really want from the theory: **the quantum amplitudes for these processes to occur**. Pay attention to the two key steps: (I) From the interaction Hamiltonian $H_I(t)$ we calculate the evolution operator, $U(t, t_0)$ using Dyson's formula (Equation 20); and (II) We expand the exponent, to get terms like $1 + H_I + H_I^2 + ...$ Each of these terms does something: For example, the term linear in H_I represent the meson decay, the term containing H_I^2 represent nucleon-nucleon scattering, etc.

We combine all these together, to calculate the quantum amplitude - and later the probability of these things to happen. Given a state, say of nucleon- anti-nucleon, we want to answer questions such as " what is the probability that they come out with a different momenta", or "what is the 1/2 lifetime of the meson", etc.

To calculate amplitudes we make an important, and slightly dodgy, assumption: **Initial and final states are eigenstates of the free theory**. This means that we take the initial state $|i\rangle$ at $t \to -\infty$, and the final state $|f\rangle$ at $t \to +\infty$, to be eigenstates of the free Hamiltonian H_0. At some level, this sounds plausible: at $t \to -\infty$, the particles in a scattering process are far separated and don't feel the effects of each other. Furthermore, we intuitively expect these states to be eigenstates of the individual number operators N, which commute with H_0, but not $H_{\rm int}$. As the particles approach each other, they interact briefly, before departing again, each going on its own merry way.

Definition. The amplitude to go from $|i\rangle$ to $|f\rangle$ is given by

$$\lim_{t_\pm \to \pm\infty} \langle f|U(t_+, t_-)|i\rangle \equiv \langle f|S|i\rangle, \tag{26}$$

where S is a **unitary operator**, known as the S-matrix. (S is for scattering).

There are a number of reasons why the assumption of non-interacting initial and final states is shaky:

- Obviously, this formalism cannot cope with bound states. For example, this formalism can't describe the scattering of an electron and proton which collide, bind, and leave as a Hydrogen atom. In fact, it is possible to circumvent this objection since it turns out that bound states show up as poles in the S-matrix.
- More importantly, a single particle, a long way from its neighbors, is never alone in field theory. This is true even in classical electrodynamics, where the electron sources

the electromagnetic field from which it can never escape. In quantum electrodynamics (QED), a related fact is that there is a cloud of virtual photons surrounding the electron. This line of thought gets us into the issues of renormalization (which will not be discussed much in this course). Nevertheless, motivated by this problem, after developing scattering theory using the assumption of non-interacting asymptotic states, we will mention a better way.

3.1. An Example: Meson Decay

Consider the relativistically normalized initial and final states,

$$\begin{aligned} |i\rangle &= \sqrt{2E_{\vec{p}}}a^{\dagger}_{\vec{p}}|0\rangle \\ |f\rangle &= \sqrt{4E_{\vec{q}_1}E_{\vec{q}_2}}b^{\dagger}_{\vec{q}_1}c^{\dagger}_{\vec{q}_2}|0\rangle. \end{aligned} \tag{27}$$

The initial state contains a single meson of momentum p; the final state contains a pair of nucleon - anti-nucleon having momentum q_1 and q_2.

We will compute now the amplitude for the decay of a meson to a nucleon - anti-nucleon pair. For that, we replace the matrix S with the exponent, $Te^{-i\int H_I(t')dt'}$, and Taylor expand the exponent. The first term is 1, and is thus not interesting. The second term in the expansion is the leading order in g, and is given by

$$\langle f|S|i\rangle = -ig\langle f|\int d^4x\psi^{\dagger}(x)\psi(x)\phi(x)|i\rangle. \tag{28}$$

(we neglect terms of $\mathcal{O}(g^2)$ and higher orders).

We proceed as follows. First, we expand $\phi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{-ip\cdot x} + a^{\dagger}_{\vec{p}}e^{+ip\cdot x}\right)$ ("Free fields", Equation 84). [Note that ϕ in Equation 28 is in the interaction picture, which is the same as the Heisenberg picture of the free theory]. Now, the a piece will turn $|i\rangle$ into something proportional to $|0\rangle$, while the $a^{\dagger}$ piece will turn $|i\rangle$ into a two meson state. However, the two meson state will have zero overlap with $\langle f|$ - and there is nothing in the ψ and $\psi^{\dagger}$ operators that lie between them to change this fact. So we have

$$\begin{aligned} \langle f|S|i\rangle &= -ig\langle f|\int d^4x\psi^{\dagger}(x)\psi(x)\int \frac{d^3k}{(2\pi)^3}\frac{\sqrt{2E_{\vec{p}}}}{\sqrt{2E_{\vec{k}}}}a_{\vec{k}}a^{\dagger}_{\vec{p}}e^{-ik\cdot x}|0\rangle \\ &= -ig\langle f|\int d^4x\psi^{\dagger}(x)\psi(x)e^{-ip\cdot x}|0\rangle, \end{aligned} \tag{29}$$

where, in the second line, we have commuted $a_{\vec{k}}$ past $a^{\dagger}_{\vec{p}}$, picking up a $\delta^{(3)}(\vec{p}-\vec{k})$ delta-function, which kills the d^3k integral.

We now similarly expand $\psi \sim b + c^\dagger$ and $\psi^\dagger \sim b^\dagger + c$. To get non-zero overlap with $\langle f|$, only the $b^\dagger$ and $c^\dagger$ contribute, for they create the nucleon and anti-nucleon from $|0\rangle$. We then have

$$\begin{aligned} \langle f|S|i\rangle &= -ig\langle 0| \int\int \tfrac{d^4x d^3k_1 d^3k_2}{(2\pi)^6} \frac{\sqrt{4E_{\vec{q}_1}E_{\vec{q}_2}}}{\sqrt{4E_{\vec{k}_1}E_{\vec{k}_2}}} c_{\vec{q}_2} b_{\vec{q}_1} c^\dagger_{\vec{k}_1} b^\dagger_{\vec{k}_2} |0\rangle e^{i(k_1+k_2-p)\cdot x} \\ &= -ig(2\pi)^4 \delta^{(4)}(q_1+q_2-p). \end{aligned} \tag{30}$$

Congratulations ! We just got our first quantum field theory amplitude.

Notice that the δ-function puts constraints on the possible decays. In particular, the decay only happens at all if $m \geq 2M$. To see this, we may always boost ourselves to a reference frame where the meson is stationary, so $p = (m, 0, 0, 0)$. Then the delta function imposes momentum conservation, namely $\vec{q}_1 = -\vec{q}_2$, and $m = 2\sqrt{M^2 + |\vec{q}|^2}$

Later we will see how to turn this quantum amplitude into something more physical, namely the lifetime of the meson. The reason this is a little tricky is that we must square the amplitude to get the probability for decay, which means we get the square of a δ-function. We will give some basic explanations below.

4. Wick's Theorem

Using Dyson's formula (Equation 20), we can calculate the time evolution operator $U(t, t_0)$. This is done using Taylor expansion, implying that we need to compute quantities like $\langle f|TH_I(x_1)...H_I(x_n)|i\rangle$, where $|i\rangle$ and $|f\rangle$ are eigenstates of the free theory. The ordering of the operators is fixed by T, time ordering. However, since the H_I's contain certain creation and annihilation operators, we really want to **move all annihilation operators to the right** where they can start killing things in $|i\rangle$. Recall that this is the definition of normal ordering. **Wick's theorem tells us how to go from time ordered products to normal ordered products.**

4.1. An Example: Recovering the Propagator

Consider a simple example: a real scalar field which we decompose in the Heisenberg picture as

$$\phi(x) = \phi^+(x) + \phi^-(x), \tag{31}$$

where

$$\begin{aligned}\phi^+(x) &= \int \tfrac{d^3p}{(2\pi)^3}\tfrac{1}{\sqrt{2E_{\vec{p}}}}a_{\vec{p}}e^{-ip\cdot x},\\ \phi^-(x) &= \int \tfrac{d^3p}{(2\pi)^3}\tfrac{1}{\sqrt{2E_{\vec{p}}}}a^\dagger_{\vec{p}}e^{+ip\cdot x},\end{aligned} \tag{32}$$

(at the moment, the $\pm$ signs on $\phi^\pm$ make little sense; they come about because $\phi^+ \sim e^{-iEt}$, which is sometimes called the positive frequency piece, while $\phi^- \sim e^{+iEt}$ is the negative frequency piece.

Choosing $x^0 > y^0$, we have

$$\begin{aligned} T\phi(x)\phi(y) &= \phi(x)\phi(y)\\ &= (\phi^+(x)+\phi^-(x))(\phi^+(y)+\phi^-(y))\\ &= \phi^+(x)\phi^+(y)+\phi^-(x)\phi^+(y)+\phi^-(y)\phi^+(x)+[\phi^+(x),\phi^-(y)]+\phi^-(x)\phi^-(y),\end{aligned} \tag{33}$$

where the last line is normal ordered: in every term (except for the commutator), all the $a_{\vec{p}}$'s are to the right of all the $a^\dagger_{\vec{p}}$. However, we did pick up the commutator - $D(x-y) = [\phi^+(x), \phi^-(y)]$ - which is the propagator we met in "Free Fields" (Equation 90). Thus, for $x^0 > y^0$ we have:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}e^{-ip\cdot(x-y)} =: \phi(x)\phi(y) : + D(x-y) \tag{34}$$

Similarly, for $y^0 > x^0$, we may repeat the calculation to find

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D(y-x) \tag{35}$$

So, combining together we have the final expression:

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \Delta_F(x-y) \tag{36}$$

where Δ_F is the Feynman propagator, defined in "Free Fields", Equation 93. Here we can start seeing why the Feynman propagator is so useful.

In "Free Fields" (Equation 99) we have derived the integral representation of the Feynman propagator,

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon} \tag{37}$$

Note: Equations 36, 37 imply that although $T\phi(x)\phi(y)$ and $:\phi(x)\phi(y):$ are both *operators*, the difference between them is a c-number function, $\Delta_F(x-y)$.

Definition. We define the **contraction** of a pair of fields in a string of operators $...\phi(x1)...\phi(x_2)...$ to mean replacing those operators with the Feynman propagator, leaving all other operators untouched. We use the notation,

$$...\overbrace{\phi(x1)...\phi(x_2)}...\tag{38}$$

to denote contraction. So, for example,

$$\overbrace{\phi(x)\phi(y)} = \Delta_F(x-y) \tag{39}$$

We have exactly similar argument for complex scalar field. We have

$$T\psi(x)\psi^\dagger(y) =: \psi(x)\psi^\dagger(y) : + \Delta_F(x-y), \tag{40}$$

which lead to defining the contraction

$$\overbrace{\psi(x)\psi^\dagger(y)} = \Delta_F(x-y) \quad \text{and} \quad \overbrace{\psi(x)\psi(y)} = \overbrace{\psi^\dagger(x)\psi^\dagger(y)} = 0 \tag{41}$$

4.2. Statement and Proof of Wick's Theorem

Wick's theorem states that **for any collection of fields,** $\phi_1 = \phi(x_1),\ \phi_2 = \phi(x_2)$, **..., we have**

$$T(\phi_1..\phi_n) =: \phi_1...\phi_n : + : \text{all possible contractions} : \tag{42}$$

Note that $\phi_1, \phi_2, ...$ are really the same field, evaluated at different points in space-time.

Lets take an example to demonstrate "all possible contractions". For $n = 4$, Equation 42 reads

$$\begin{aligned} T(\phi_1\phi_2\phi_3\phi_4) =: \phi_1\phi_2\phi_3\phi_4 : \quad &+ \quad \overbrace{\phi_1\phi_2} : \phi_3\phi_4 : + \overbrace{\phi_1\phi_3} : \phi_2\phi_4 : + \text{(4 similar terms)} \\ &+ \quad \overbrace{\phi_1\phi_2}\,\overbrace{\phi_3\phi_4} + \overbrace{\phi_1\phi_3}\,\overbrace{\phi_2\phi_4} + \overbrace{\phi_1\phi_4}\,\overbrace{\phi_2\phi_3}\,. \end{aligned} \tag{43}$$

Proof. The proof of Wick's theorem proceeds by induction and a little thought. It's true for $n = 2$. Suppose it's true for $\phi_2...\phi_n$. We now add ϕ_1. We will take $x_1^0 > x_k^0$ for all $k = 2, ..., n$. Then we can pull ϕ_1 out to the left of the time ordered product, writing

$$T(\phi_1\phi_2...\phi_n) = (\phi_1^+ + \phi_1^-)(: \phi_2...\phi_n : + : \text{contractions} :). \tag{44}$$

The ϕ_1^- term stays where it is since it is already normal ordered. But in order to write the right-hand side as a normal ordered product, the ϕ_1^+ term has to make its way past the crowd of ϕ_k^- operators. Each time it moves past ϕ_k^-, we pick up a factor of $\overbrace{\phi_1\phi_k} = \Delta_F(x_1 - x_k)$ from the commutator.

4.3. An Example: Nucleon Scattering

Let us look at $\psi\psi \to \psi\psi$ scattering. We are interested in calculating the cross section for the scattering; In practice, we calculate here the amplitude, and then we will square it to get the cross section.

We have the initial and final states:

$$\begin{aligned}|i\rangle &= \sqrt{2E_{\vec{p_1}}}\sqrt{2E_{\vec{p_2}}}b^\dagger_{\vec{p_1}}b^\dagger_{\vec{p_2}}|0\rangle \equiv |\vec{p_1},\vec{p_2}\rangle \\ |f\rangle &= \sqrt{2E_{\vec{p_1}'}}\sqrt{2E_{\vec{p_2}'}}b^\dagger_{\vec{p_1}'}b^\dagger_{\vec{p_2}'}|0\rangle \equiv |\vec{p_1}',\vec{p_2}'\rangle.\end{aligned} \tag{45}$$

We can look at the expansion of $\langle f|S|i\rangle$. In fact, we are only interested in $\langle f|S-1|i\rangle$, since we are not interested in situations without scattering (namely, the two outgoing particles have the same momenta as the two incoming ones).

The first order in g is not going to give us anything, since it just creates a meson (recall that we are in the framework of Yukawa theory, whose Lagrangian is given in Equation 7). At order g^2 we have the term

$$\frac{(-ig)^2}{2}\int d^4x_1 d^4x_2 T\left(\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\right). \tag{46}$$

The interaction Hamiltonian is evaluated at two different points, x_1 and x_2, but we have to time-order it. We can use Wick's theorem, which tells us that the time ordered component is equal to the normal ordered component + all contractions. There will be many components; however, most of them will give 0. This is because terms with, e.g., ϕ to the left destroy a meson; but since no meson initially exists, their contribution to the Hamiltonian is 0.

We find that only one term in the string of operators, which looks like

$$:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):\overbrace{\phi(x_1)\phi(x_2)} \tag{47}$$

contributes to a non-zero Hamiltonian. This term will contribute to the scattering because the two ψ fields annihilate the ψ particles, while the two $\psi^\dagger$ fields create ψ particles. Any other way of ordering the ψ and $\psi^\dagger$ fields will give zero contribution. This means that we have

$$\begin{aligned}&\langle p_1', p_2'|:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):|p_1,p_2\rangle \times \overbrace{\phi(x_1)\phi(x_2)} \\ =\ & \langle p_1', p_2'|\psi^\dagger(x_1)\psi^\dagger(x_2)|0\rangle\langle 0|\psi(x_1)\psi(x_2)|p_1,p_2\rangle \times \overbrace{\phi(x_1)\phi(x_2)} \\ =\ & \left(e^{ip_1'\cdot x_1+ip_2'\cdot x_2}+e^{ip_1'\cdot x_2+ip_2'\cdot x_1}\right)\left(e^{-ip_1\cdot x_1-ip_2\cdot x_2}+e^{-ip_1\cdot x_2-ip_2\cdot x_1}\right)\times\overbrace{\phi(x_1)\phi(x_2)} \\ =\ & \left[e^{ix_1\cdot(p_1'-p_1)+ix_2\cdot(p_2'-p_2)}+e^{ix_1\cdot(p_2'-p_1)+ix_2\cdot(p_1'-p_2)}+(x_1\leftrightarrow x_2)\right]\times\overbrace{\phi(x_1)\phi(x_2)}\end{aligned} \tag{48}$$

where, in going to the third line, we have used the fact that for relativistically normalized states,

$$\langle 0|\psi(x)|p\rangle = e^{-ip\cdot x}. \tag{49}$$

We can insert now the result in Equation 48 into Equation 46, to get the expression for $\langle f|S|i\rangle$ at order g^2:

$$\frac{(-ig)^2}{2}\int d^4x_1 d^4x_2 \left[e^{i\ldots} + e^{i\ldots} + (x_1 \leftrightarrow x_2)\right] \int \frac{d^4k}{(2\pi)^4}\frac{ie^{ik\cdot(x_1-x_2)}}{k^2-m^2+i\epsilon}. \tag{50}$$

where the final integral is the ϕ propagator which comes from the contraction in Equation 47. The $(x_1 \leftrightarrow x_2)$ terms double up with the others to cancel the factor of 1/2 out front. Meanwhile, the x_1 and x_2 integrals give delta-functions. We are left with the expression

$$\begin{aligned}(-ig)^2 \int \tfrac{d^4k}{(2\pi)^4}\tfrac{i(2\pi)^8}{k^2-m^2+i\epsilon} \quad &\big[\delta^{(4)}(p_1{}' - p_1 + k)\delta^{(4)}(p_2{}' - p_2 - k) \\ &+ \delta^{(4)}(p_2{}' - p_1 + k)\delta^{(4)}(p_1{}' - p_2 - k)\big]\end{aligned} \tag{51}$$

Finally, we can trivially do the d^4k integral using the delta-functions to get

$$i(-ig)^2\left[\frac{1}{(p_1-p_1{}')^2 - m^2 + i\epsilon} + \frac{1}{(p_1-p_2{}')^2 - m^2 + i\epsilon}\right](2\pi)^4\delta^{(4)}(p_1+p_2-p_1{}'-p_2{}')$$

In fact, for this process we can drop the $+i\epsilon$ terms since the denominator is never zero (we do have singularities in particle - anti-particle ($\psi\bar{\psi}$) interactions, but not in $\psi - \psi$ scattering). To see this, we can go to the center of mass frame, where $\vec{p}_1 = -\vec{p}_2$ and, by momentum conservation, $|\vec{p}_1| = |\vec{p}_1{}'|$. This ensures that the 4-momentum of the meson is $k = (0, \vec{p} - \vec{p}')$, so $k^2 < 0$. We can therefore write the end result,

$$i(-ig)^2\left[\frac{1}{(p_1-p_1{}')^2 - m^2} + \frac{1}{(p_1-p_2{}')^2 - m^2}\right](2\pi)^4\delta^{(4)}(p_1+p_2-p_1{}'-p_2{}') \tag{52}$$

We will shortly see another, much simpler way to reproduce this result using Feynman diagrams. This will also shed light on the physical interpretation.

Another Example: Meson-Nucleon Scattering

If we want to compute $\psi\phi \to \psi\phi$ scattering at order g^2, we would need to pick out the term

$$:\psi^\dagger(x_1)\phi(x_1)\psi(x_2)\phi(x_2):\overbrace{\psi(x_1)\psi^\dagger(x_2)}, \tag{53}$$

and a similar term with ψ and $\psi^\dagger$ exchanged. Once more, this term also contributes to similar scattering processes, including $\bar{\psi}\phi \to \bar{\psi}\phi$ and $\phi\phi \to \psi\bar{\psi}$.

5. Feynman Diagrams

As the above example demonstrates, to actually compute scattering amplitudes using Wicks theorem is rather tedious. Fortunately, there is an equivalent, and a much better way due to **Feynman**, that also involves drawing nice pictures. These pictures represent the expansion of $\langle f|S|i\rangle$ and we will learn how to associate numbers (or at least integrals) to them. These pictures are called **Feynman diagrams**.

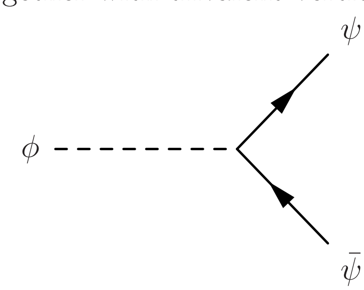
The object that we really want to compute is $\langle f|S-1|i\rangle$, since we are not interested in processes where no scattering occurs. We can draw diagrams, in which each object has a 1-1 correspondence with the terms in the expansion of $\langle f|S-1|i\rangle$.

The various terms in the perturbative expansion can be represented pictorially as follows:

- For each particle in the initial state $|i\rangle$ and each particle in the final state $|f\rangle$, we draw an **external line**. We will use the following convention: *dotted lines* for mesons, and *solid lines* for nucleons.

  Next, (1) assign a directed momentum p to each line; (2) add an arrow to solid lines to denote charge; we will choose an **incoming (outgoing)** arrow in the initial state for ψ ($\bar{\psi}$). We choose the **reverse convention** for the final state, where an outgoing arrow denotes ψ.

- Join the external lines together with trivalent vertices (see Figure below).



5.1. Feynman Rules

To each diagram we associate a number, using the *Feynman rules.*

- Add a momentum k to each internal line (different k to different lines !).

- To each vertex, write down a factor of

$$(-ig)(2\pi)^4\delta^{(4)}\left(\sum_i k_i\right), \tag{54}$$

where $\sum_i k_i$ is the sum of all momenta flowing **into** the vertex.

- For each internal dotted line, corresponding to a ϕ particle with momentum k, we write down a factor of
$$\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon} \tag{55}$$
The same factor is included for solid internal ψ lines, with m replaced by the nucleon mass, M.

6. Examples of Scattering Amplitudes

Let's apply the Feynman rules to compute the amplitudes for various processes. We start with something familiar:

Nucleon Scattering Revisited

Let us consider again the $\psi\psi \to \psi\psi$ scattering at order g^2, discussed in section 4.3 (Equation 46). We can draw the two simplest diagrams contributing to this process. They are shown in Figure 1.[1]

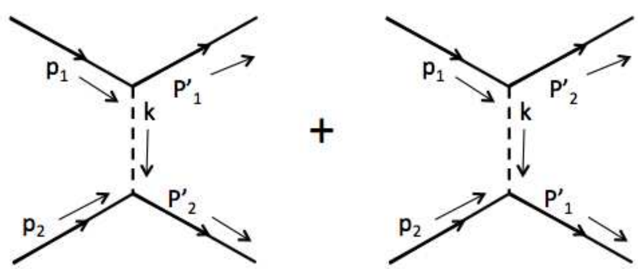


Fig. 1.— The lowest order Feynman diagrams for nucleon scattering

Applying the Feynman rules to these diagrams, we get

$$\begin{aligned}(-ig)^2 \int \tfrac{d^4k}{(2\pi)^4}\tfrac{i}{k^2-m^2+i\epsilon}(2\pi)^8 &\left[\delta^{(4)}(p_1-p_1'-k)\delta^{(4)}(p_2+k-p_2')\right] \\ &+(\text{same}) \times \left[\delta^{(4)}(p_1-p_2'-k)\delta^{(4)}(p_2+k-p_1')\right] \\ = i(-ig)^2 &\left[\tfrac{1}{(p_1-p_1')^2-m^2} + \tfrac{1}{(p_1-p_2')^2-m^2}\right](2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-p_2').\end{aligned} \tag{56}$$

---

[1]The left diagram is known as "T-channel", while the right diagram is known as "U-channel". The diagram in Figure 4 (right) is known as "S-channel".

The first line comes from the left diagram in Figure 1, while the second line comes from the right diagram. The $(-ig)^2$ terms come from the two vortices, while the $\int \frac{d^4k}{(2\pi)^4}$ arise from the internal line. The δ-functions enable to calculate the integral over d^4k, and obtain the last line. This last line is clearly in agreement with the calculation in Equation 51 that we performed earlier.

There is a nice physical interpretation of these diagrams. We talk, rather loosely, of the nucleons exchanging a meson which, in the first diagram, has momentum $k = (p_1 - p_1') = (p_2 - p_2')$. This meson doesn't satisfy the usual energy dispersion relation, because $k^2 \neq m^2$: the meson is called a **virtual particle** and is said to be **off-shell** (or, sometimes, off mass-shell). Heuristically, it can't live long enough for its energy to be measured to great accuracy. In contrast, the momentum on the external, nucleon legs all satisfy $p^2 = M^2$, the mass of the nucleon. They are **on-shell**. One final note: the addition of the two diagrams above ensures that the particles satisfy Bose statistics.

There are also more complicated diagrams which will contribute to the scattering process at higher orders. For example, we have the two diagrams shown in Figure 2, and similar diagrams with p_1' and p_2' exchanged. Using the Feynman rules, each of these diagrams translates into an integral that we will not attempt to calculate here. And so we go on, with increasingly complicated diagrams, all appearing at higher order in the coupling constant g.

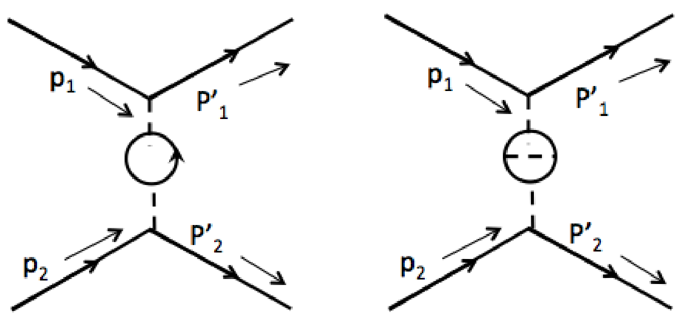


Fig. 2.— Left: a contribution at $\mathcal{O}(g^4)$; Right: a contribution at $\mathcal{O}(g^6)$.

Amplitudes

Our final result for the nucleon scattering amplitude $\langle f|S-1|i\rangle$ at order g^2 was

$$i(-ig)^2 \left[\frac{1}{(p_1 - p_1')^2 - m^2} + \frac{1}{(p_1 - p_2')^2 - m^2}\right](2\pi)^4\delta^{(4)}(p_1 + p_2 - p_1' - p_2')$$

The δ-function follows from the conservation of 4-momentum which, in turn, follows from spacetime translational invariance. It is common to all S-matrix elements. We **define the**

amplitude $\mathcal{A}_{fi}$ by stripping off this momentum-conserving delta-function,

$$\langle f|S-1|i\rangle = i\mathcal{A}_{fi}(2\pi)^4\delta^{(4)}(p_F - p_I) \tag{57}$$

where p_I (p_F) is the sum of the initial (final) 4-momenta, and the factor of i out front is a convention which is there to match non-relativistic quantum mechanics. We can now refine our Feynman rules to compute the amplitude $i\mathcal{A}_{fi}$ itself:

- Draw all possible diagrams with appropriate external legs and impose 4-momentum conservation at each vertex.
- Write down a factor of $(-ig)$ at each vertex.
- For each internal line, write down the propagator.
- Integrate over momentum k flowing through each loop $\int d^4k/(2\pi)^4$.

This last step deserves a short explanation. The diagrams we have computed so far are the leading order diagrams of this process, and thus have no loops. They are thus called **tree level** diagrams. It's not hard to convince yourself that in tree diagrams, momentum conservation at each vertex is sufficient to determine the momentum flowing through each internal line. As opposed to that, for **loop diagrams**, such as those shown in Figure 2, this is no longer the case.

6.1. Other processes

Nucleon to Meson Scattering.

Let's now look at the amplitude for a nucleon-anti-nucleon pair to annihilate into a pair of mesons: $\psi\bar{\psi} \to \phi\phi$. The simplest Feynman diagrams for this process are shown in Figure 3 where the virtual particle in these diagrams is now the nucleon ψ rather than the meson ϕ. This fact is reflected in the denominator of the amplitudes which are given by

$$i\mathcal{A} = (-ig)^2\left[\frac{i}{(p_1-p_1')^2 - M^2} + \frac{i}{(p_1-p_2')^2 - M^2}\right]. \tag{58}$$

As in Equation 52, we have dropped the $i\epsilon$ from the propagators, as the denominator never vanishes.

Nucleon - Anti-Nucleon Scattering

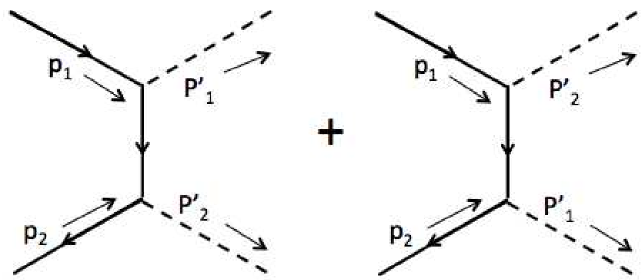


Fig. 3.— The two lowest order Feynman diagrams for nucleon to meson scattering

For the scattering of a nucleon and an anti-nucleon, $\psi\bar{\psi} \to \psi\bar{\psi}$, the Feynman diagrams are a little different. At lowest order, they are given by the diagrams of Figure 4. It is a simple matter to write down the amplitude using the Feynman rules,

$$i\mathcal{A} = (-ig)^2 \left[\frac{1}{(p_1 - p_1')^2 - m^2} + \frac{i}{(p_1 + p_2)^2 - m^2 + i\epsilon} \right]. \tag{59}$$

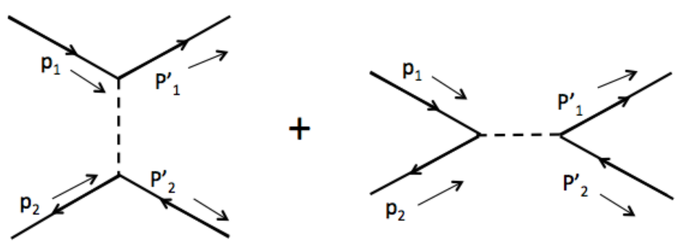


Fig. 4.— The two lowest order Feynman diagrams for nucleon - anti-nucleon scattering

Notice that the momentum dependence in the second term is different from that of nucleon-nucleon scattering (Equation 56), reflecting the different Feynman diagram that contributes to the process. In the center of mass frame, $\vec{p}_1 = -\vec{p}_2$, the denominator of the second term is $4(M^2 + \vec{p}_1^2) - m^2$. If $m < 2M$, then this term never vanishes and we may drop the $i\epsilon$. In contrast, if $m > 2M$, then the amplitude corresponding to the second diagram diverges at some value of $\vec{p}$. In this case it turns out that we may also neglect the $i\epsilon$ term, although for a different reason: the meson is unstable when $m > 2M$, a result we derived earlier (see Equation 30). When correctly treated, this instability adds a finite imaginary piece to the denominator which overwhelms the $i\epsilon$.

Nonetheless, the increase in the scattering amplitude which we see in the second diagram when $4(M^2 + \vec{p}_1^2) = m^2$ is what allows us to discover new particles: they appear as a resonance in the cross section. For example, the Figure 5 shows the cross-section (roughly the amplitude squared) plotted vertically for $e^+e^- \to \mu^+\mu^-$ scattering from the ALEPH experiment in CERN. The horizontal axis shows the center of mass energy. The curve rises sharply around 91 GeV, the mass of the Z-boson.

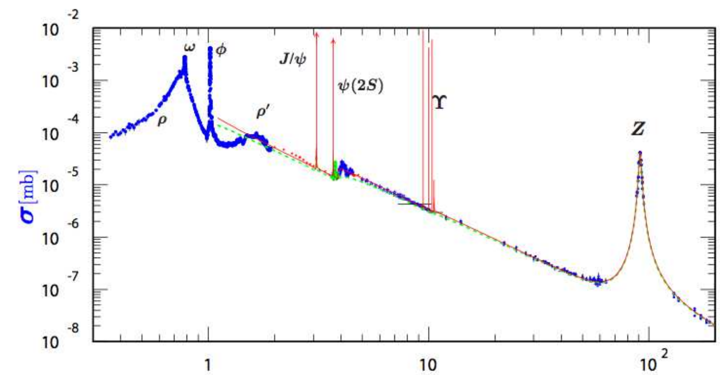


Fig. 5.— Cross section of e^+e^- collision, as measured by ALEPH detector in CERN (ref: PDG, 2012).

Meson Scattering

For $\phi\phi \to \phi\phi$, the simplest diagram we can write down has a single loop, and momentum conservation at each vertex is no longer sufficient to determine every momentum passing through the diagram. We choose to assign the single undetermined momentum k to the right-hand propagator. All other momenta are then determined. The amplitude corresponding to the diagram shown in figure 6 is

$$i\mathcal{A} = (-ig)^4 \int \frac{d^4k}{(2\pi)^4} \quad \frac{1}{(k^2-M^2+i\epsilon)((k+p_1')^2-M^2+i\epsilon)} \times \frac{1}{((k+p_1'-p_1)^2-M^2+i\epsilon)((k-p_2')^2-M^2+i\epsilon)}$$

These integrals can be tricky. For large k, this integral goes as $\int d^4k/k^8$, which is at least convergent as $k \to \infty$. But this won't always be the case.

6.2. Mandelstam Variables

We see that in many of the amplitudes above - in particular those that include the exchange of just a single particle - the same combinations of momenta are appearing frequently in the denominators. There are standard names for various sums and differences of

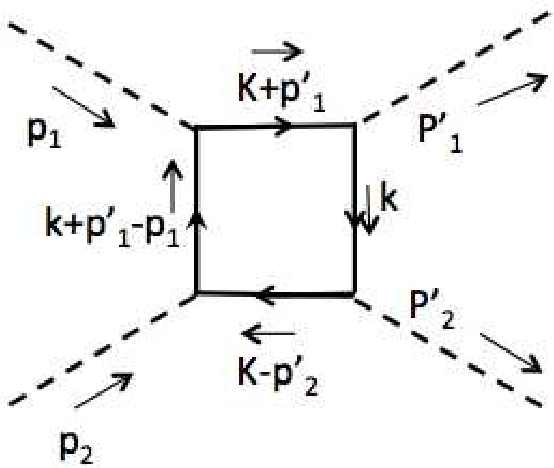


Fig. 6.— The lowest order Feynman diagram for meson - meson scattering

momenta: they are known as **Mandelstam variables**. They are

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p'_1 + p'_2)^2, \\ t &= (p_1 - p'_1)^2 = (p_2 - p'_2)^2, \\ u &= (p_1 - p'_2)^2 = (p_2 - p'_1)^2, \end{aligned} \tag{60}$$

where, as in the examples above, p_1 and p_2 are the momenta of the two initial particles, and p'_1 and p'_2 are the momenta of the final two particles.

We can define these variables whether the particles involved in the scattering are the same or different. To get a feel for what these variables mean, let's assume all four particles are the same. We sit in the center of mass frame, so that the initial two particles have four-momenta

$$p_1 = (E, 0, 0, p) \text{ and } p_2 = (E, 0, 0, -p). \tag{61}$$

The particles then scatter at some angle θ, and leave with momenta

$$p'_1 = (E, 0, p\sin\theta, p\cos\theta) \text{ and } p'_2 = (E, 0, -p\sin\theta, -p\cos\theta). \tag{62}$$

Then from the above definitions, we find

$$s = 4E^2 \text{ and } t = -2p^2(1 - \cos\theta) \text{ and } u = -2p^2(1 + \cos\theta). \tag{63}$$

The variable s measures the **total center of mass energy of the collision**, while the variables t and u are measures of the **momentum exchanged between particles**. (They are basically equivalent, just with the outgoing particles swapped around).

Now the amplitudes that involve exchange of a single particle can be written simply in terms of the Mandelstam variables. For example, for nucleon-nucleon scattering, the amplitude in Equation 56 is schematically $\mathcal{A} \sim (t - m^2)^{-1} + (u - m^2)^{-1}$. For the nucleon-anti-nucleon scattering, the amplitude (Equation 59) is $\mathcal{A} \sim (t-m^2)^{-1} + (s-m^2)^{-1}$. We say that the first case involves "t-channel" and "u-channel" diagrams. Meanwhile the nucleon - anti-nucleon scattering is said to involve "t-channel" and "s-channel" diagrams. (The first diagram indeed includes a vertex that looks like the letter "T", with some imagination).

Note that there is a relationship between the Mandelstam variables. When all the masses are the same we have $s + t + u = 4M^2$. When the masses of all 4 particles differ, this becomes $s + t + u = \sum_i M_i^2$.

6.3. The Yukawa Potential

So far, we have computed the quantum amplitudes for various scattering processes. But these quantities are a little abstract. In Section 7 below we will see how to turn amplitudes into measurable quantities such as cross-sections, or the lifetimes of unstable particles. Here we will instead show how to translate the amplitude (in Equation 52) for nucleon scattering into something familiar from Newtonian mechanics: a potential, or **force**, between the particles.

Let's start by asking a simple question in classical field theory that will turn out to be relevant. Suppose that we have a fixed δ-function source for a real scalar field ϕ, that persists for all time. What is the profile of $\phi(\vec{x})$? To answer this, we must solve the static Klein-Gordon equation,

$$-\nabla^2\phi + m^2\phi = \delta^{(3)}(\vec{x}). \tag{64}$$

We can solve this using the Fourier transform,

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{\phi}(\vec{k}). \tag{65}$$

Plugging this into Equation 64, we find $(\vec{k}^2 + m^2)\tilde{\phi}(\vec{k}) = 1$, giving us the solution

$$\phi(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2 + m^2}. \tag{66}$$

Let us now do this integral. Changing to polar coordinates, and writing $\vec{k}\cdot\vec{x} = kr\cos\theta$, we have

$$\begin{aligned} \phi(\vec{x}) &= \tfrac{1}{(2\pi)^2}\int_0^\infty dk \tfrac{k^2}{k^2+m^2}\tfrac{2\sin kr}{kr} \\ &= \tfrac{1}{(2\pi)^2 r}\int_{-\infty}^{\infty} dk \tfrac{k\sin kr}{k^2+m^2} \\ &= \tfrac{1}{2\pi r}\mathrm{Re}\left[\int_{-\infty}^{\infty}\tfrac{dk}{2\pi i}\tfrac{ke^{ikr}}{k^2+m^2}\right]. \end{aligned} \tag{67}$$

We compute this last integral by closing the contour in the upper half plane, $k \to +i\infty$, picking up the pole at $k = +im$. This gives

$$\phi(\vec{x}) = \frac{1}{4\pi r}e^{-mr}. \tag{68}$$

The field thus dies off exponentially quickly at distances $1/m$, the Compton wavelength of the meson.

Now we understand the profile of the ϕ field, what does this have to do with the force between ψ particles? We do very similar calculations to that above in electrostatics where a charged particle acts as a δ-function source for the gauge potential: $-\nabla^2 A_0 = \delta^{(3)}(\vec{x})$, which is solved by $A_0 = 1/4\pi r$. The profile for A_0 then acts as the potential energy for another charged (test) particle moving in this background. Can we give the same interpretation to our scalar field? In other words, is there a classical limit of the scalar Yukawa theory where the ψ particles act as δ-function sources for ϕ, creating the profile in Equation 68? And, if so, is this profile then felt as a static potential? The answer is essentially yes, at least in the limit $M \gg m$. But the correct way to describe the potential felt by the ψ particles is not to talk about classical fields at all, but instead work directly with the quantum amplitudes.

Our strategy is to compare the nucleon scattering amplitude (Equation 52) to the corresponding amplitude in non-relativistic quantum mechanics for two particles interacting through a potential. To make this comparison, we should first take the non-relativistic limit of Equation 52. Let's work in the center of mass frame, with $\vec{p} \equiv \vec{p}_1 = -\vec{p}_2$ and $\vec{p}' \equiv \vec{p}'_1 = -\vec{p}'_2$. The non-relativistic limit means $|\vec{p}| \ll M$ which, by momentum conservation, ensures that $|\vec{p}'| \ll M$. In fact one can check that, for this particular example, this limit doesn't change the scattering amplitude in Equation 52: it is given by

$$i\mathcal{A} = +ig^2\left[\frac{1}{(p-p')^2+m^2} + \frac{1}{(p+p')^2+m^2}\right]. \tag{69}$$

How do we compare this to scattering in quantum mechanics? Consider two particles, separated by a distance $\vec{r}$, interacting through a potential $U(\vec{r})$. In non-relativistic quantum mechanics, the amplitude for the particles to scatter from momentum states $\pm\vec{p}$ into momentum states $\pm\vec{p}'$ can be computed in perturbation theory, using the techniques described in Section 1 above. To leading order, known in this context as the **Born approximation**, the amplitude is given by

$$\langle \vec{p}'|U(\vec{r})|\vec{p}\rangle = -i\int d^3r U(\vec{r})e^{-i(\vec{p}-\vec{p}')\cdot\vec{r}}. \tag{70}$$

There is a relative factor of $(2M)^2$ that arises in comparing the quantum field theory amplitude $\mathcal{A}$ to $\langle \vec{p}'|U(\vec{r})|\vec{p}\rangle$, that can be traced to the relativistic normalization of the states $|p_1, p_2\rangle$. (It is also necessary to get the dimensions of the potential to work out correctly). Including this factor, and equating the expressions for the two amplitudes, we get

$$\int d^3r U(\vec{r})e^{-i(\vec{p}-\vec{p}')\cdot\vec{r}} = \frac{-\lambda^2}{(\vec{p}-\vec{p}')^2 + m^2}, \tag{71}$$

where we introduced the dimensionless parameter, $\lambda = g/2M$. We can invert this to find

$$U(\vec{r}) = -\lambda^2 \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\vec{p}\cdot\vec{r}}}{\vec{p}^2 + m^2}. \tag{72}$$

But this is exactly the integral in Equation 66, which we just did in classical theory!. We thus have

$$U(\vec{r}) = \frac{-\lambda^2}{4\pi r}e^{-mr}, \tag{73}$$

where $\vec{r}$ is the separation between the two particles, and $\lambda = g/2M$. This is **Yukawa potential** (suggested by Yukawa for the strong force). The force has a range $1/m$, the Compton wavelength of the exchanged particle. The minus sign tells us that the potential is attractive.

Notice that quantum field theory has given us an entirely new perspective on the nature of forces between particles. *Rather than being a fundamental concept, the force arises from the virtual exchange of other particles*, in this case the meson. We will show below how the Coulomb force arises from quantum field theory due to the exchange of virtual photons.

We could repeat the calculation for nucleon-anti-nucleon scattering. The amplitude from field theory is given in Equation 59. The first term in this expression gives the same result as for nucleon-nucleon scattering *with the same sign.* The second term vanishes in the non-relativistic limit (it is an example of an interaction that doesn't have a simple Newtonian interpretation). There is no longer a factor of 1/2 in Equation 70, because the incoming/outgoing particles are not identical, so we learn that the potential between a nucleon and anti-nucleon is again given by Equation 73. This reveals a key feature of forces arising due to the exchange of scalars: **they are universally attractive**. Notice that this is different from forces due to the exchange of a spin 1 particle - such as electromagnetism - where the sign flips when we change the charge. However, for forces due to the exchange of a spin 2 particle (i.e., gravity) the force is again universally attractive.

6.4. Feynman rules for ϕ^4 Theory

We now look at another weakly coupled theory, the ϕ^4 theory, whose Lagrangian was given in Equation 5, with $\lambda \ll 1$. Let us briefly look at the Feynman rules and scattering amplitudes for the interaction Hamiltonian

$$H_{\text{int}} = \frac{\lambda}{4!}\phi^4 \tag{74}$$

The theory now has a single interaction vertex, which comes with a factor of $(-i\lambda)$, while the other Feynman rules remain the same (see Figure 7). Note that we assign $(-i\lambda)$ to the vertex, rather than $(-i\lambda/4!)$. To see why this is, we can look at $\phi\phi \to \phi\phi$ scattering, which has its lowest contribution at order λ, with the term

$$\frac{-i\lambda}{4!}\langle p_1' p_2'| : \phi(x)\phi(x)\phi(x)\phi(x) : |p_1, p_2\rangle. \tag{75}$$

Any one of the fields can do the job of annihilation or creation. This gives 4! different contractions, which cancels the 1/4! sitting out front.

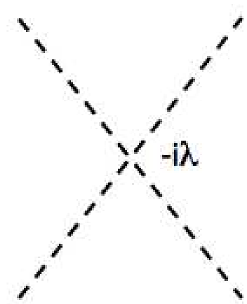


Fig. 7.— Feynman diagram for ϕ^4 theory

Feynman diagrams in the ϕ^4 theory sometimes come with extra combinatoric factors (typically 2 or 4) which are known as symmetry factors that one must take into account. For more details, see the book by Peskin and Schroeder.

Using the Feynman rules, the scattering amplitude for $\phi\phi \to \phi\phi$ is simply $i\mathcal{A} = -i\lambda$. Note that it doesn't depend on the angle at which the outgoing particles emerge: in ϕ^4 theory the leading order two-particle scattering occurs with equal probability in all directions. Translating this into a potential between two mesons, we have

$$U(\vec{r}) = \frac{\lambda}{(2m)^2}\int \frac{d^3p}{(2\pi)^3} r^{+i\vec{p}\cdot\vec{r}} = \frac{\lambda}{(2m)^2}\delta^{(3)}(\vec{r}), \tag{76}$$

So scattering in ϕ^4 theory is due to a δ-function potential. The particles don't know what hit them until it's over.

6.5. Connected Diagrams and Amputated Diagrams

We have seen how one can compute scattering amplitudes by writing down all Feynman diagrams and assigning integrals to them using the Feynman rules. In fact, there are a couple of caveats about what Feynman diagrams you should write down. Both of these caveats are related to the assumption we made earlier that "initial and final states are eigenstates of the free theory" which, as we mentioned at the time, is not strictly accurate. The two caveats which go some way towards ameliorating the problem are the following.

- We consider only connected Feynman diagrams, where every part of the diagram is connected to at least one external line. As we shall see shortly, this will be related to the fact that the vacuum $|0\rangle$ of the free theory is not the true vacuum $|\Omega\rangle$ of the interacting theory. An example of a diagram that is not connected is shown in Figure 8.
- We do not consider diagrams with loops on external lines, for example the diagram shown in the Figure 8, right. We will not explain how to take these into account in this course, but on a more advanced course. They are related to the fact that the one-particle states of the free theory are not the same as the one-particle states of the interacting theory. In particular, correctly dealing with these diagrams will account for the fact that particles in interacting quantum field theories are never alone, but surrounded by a cloud of virtual particles. We will refer to diagrams in which all loops on external legs have been cut-off as "amputated".

7. What we measure: Cross Sections and Decay Rates

So far we have learn to compute the quantum amplitudes for particles decaying or scattering. As usual in quantum theory, the probabilities for things to happen are the (modulus) square of the quantum amplitudes. In this section we will compute these probabilities, known as decay rates and cross sections. One small subtlety here is that the S-matrix elements $\langle f|S-1|i\rangle$ all come with a factor of $(2\pi)^4\delta^{(4)}(p_F-p_I)$, so we end up with the square of a delta-function. As we will now see, this comes from the fact that we are working in an infinite space.

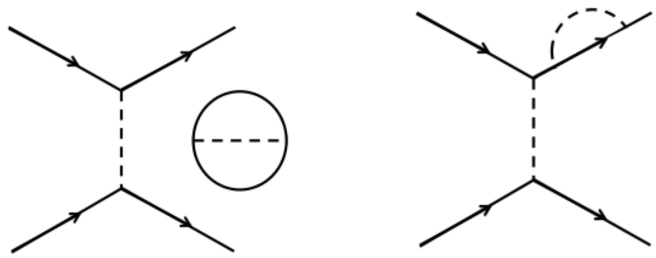


Fig. 8.— Left: a disconnected diagram; Right: An un-amputated diagram.

7.1. Fermi's Golden Rule

Let's start with something familiar and recall how to derive Fermi's golden rule from Dyson's formula (Equation 20). For two energy eigenstates $|m\rangle$ and $|n\rangle$, with $E_m \neq E_n$, we consider only the leading order in the interaction,

$$\begin{aligned} \langle m|U(t)|n\rangle &= -i\langle m| \int_0^t dt H_I(t)|n\rangle \\ &= -i\langle m|H_{\text{int}}(t)|n\rangle \int_0^t dt' e^{i\omega t'} \\ &= -\langle m|H_{\text{int}}(t)|n\rangle \frac{e^{i\omega t}-1}{\omega} \end{aligned} \tag{77}$$

where $\omega = E_m - E_n$. This gives us the probability for the transition from $|n\rangle$ to $|m\rangle$ in time t, as

$$P_{n\to m}(t) = |\langle m|U(t)|n\rangle|^2 = 2|\langle m|H_{\text{int}}|n\rangle|^2 \left(\frac{1-\cos\omega t}{\omega^2}\right) \tag{78}$$

The function in brackets is plotted in Figure 9 for fixed t. We see that in time t, most transitions happen in a region between energy eigenstates separated by $\Delta E = 2\pi/t$. As $t \to \infty$, the function in the figure starts to approach a delta-function. To find the normalization, we can calculate

$$\begin{aligned} &\int_{-\infty}^{+\infty} d\omega \left(\tfrac{1-\cos\omega t}{\omega^2}\right) = \pi t \\ \Rightarrow \quad &\left(\tfrac{1-\cos\omega t}{\omega^2}\right) \to \pi t\delta(\omega) \quad \text{as} \quad t\to\infty. \end{aligned}$$

Consider now a transition from state $|n\rangle$ to a mixture of states with density $\rho(E)$. In the limit $t \to \infty$, we get the transition probability

$$\begin{aligned} P_{n\to m} &= \int dE_m \rho(E_m) 2|\langle m|H_{\text{int}}|n\rangle|^2 \left(\tfrac{1-\cos\omega t}{\omega^2}\right) \\ &\to 2\pi|\langle m|H_{\text{int}}|n\rangle|^2 \rho(E) t \end{aligned} \tag{79}$$

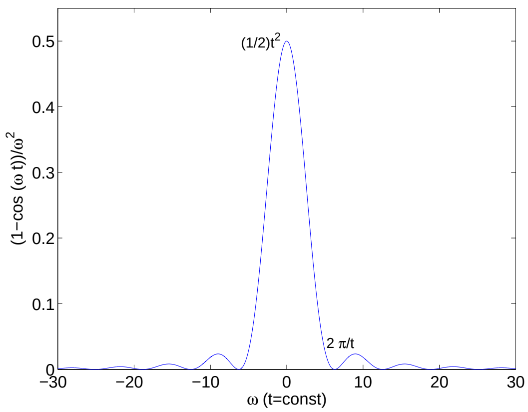


Fig. 9.— The function in bracket in Equation 78, plotted at a fixed time.

which gives a constant probability for the transition per unit time for states around the same energy $E_n \sim E_m = E$.

$$\dot{P}_{n\to m} = 2\pi |\langle m|H_{\rm int}|n\rangle|^2 \rho(E) \tag{80}$$

This is known as **Fermi's Golden Rule**.

In the above derivation, we were fairly careful with taking the limit as $t \to \infty$. Suppose we were a little sloppier, and first chose to compute the amplitude for the state $|n\rangle$ at $t \to -\infty$ to transition to the state $|m\rangle$ at $t \to +\infty$. Then we get

$$-i\langle m| \int_{t=-\infty}^{t=+\infty} H_I(t)|n\rangle = -i\langle m|H_{\rm int}|n\rangle 2\pi\delta(\omega). \tag{81}$$

Now when squaring the amplitude to get the probability, we run into the problem of the square of the delta-function: $P_{n\to m} = |\langle m|H_{\rm int}|n\rangle|^2 (2\pi)^2 \delta(\omega)^2$. Tracking through the previous computations, we realize that the extra infinity is coming because $P_{n\to m}$ is the probability for the transition to happen in infinite time $t \to \infty$. We can write the delta-functions as

$$(2\pi)^2 \delta(\omega)^2 = (2\pi)\delta(\omega)T, \tag{82}$$

where T is shorthand for $t \to \infty$ (this is a very similar trick to the one we used when looking at the vacuum energy, in "Free fields", Equation 25).

We now divide by this power of T to get the transition probability per unit time,

$$\dot{P}_{n\to m} = 2\pi |\langle m|H_{\rm int}|n\rangle|^2 \delta(\omega), \tag{83}$$

which, after integrating over the density of final states, gives us back Fermi's Golden rule. The reason that we have stressed this point is because, in our field theory calculations, we have computed the amplitudes in the same way as in Equation 81, and the square of the $\delta^{(4)}$-functions will just be re-interpreted as spacetime volume factors.

7.2. Decay Rates

Let's now look at the probability for a single particle $|i\rangle$ of momentum p_I ($I =$ initial) to decay into some number of particles $|f\rangle$ with momentum p_i and total momentum $p_F = \sum_i p_i$. This is given by

$$P = \frac{|\langle f|S|i\rangle|^2}{\langle f|f\rangle\langle i|i\rangle}. \tag{84}$$

The states obey the relativistic normalization formula ("Free fields", Equation 65),

$$\langle i|i\rangle = (2\pi)^3 2E_{\vec{p}_I}\delta^{(3)}(0) = 2E_{\vec{p}_I}V, \tag{85}$$

where we have replaced the $\delta^{(3)}(0)$ by the volume of the 3-space. Similarly,

$$\langle f|f\rangle = \prod_{\text{final states}} 2E_{\vec{p}_i}V \tag{86}$$

If we place our initial particle at rest, so $\vec{p}_I = 0$ and $E_{\vec{p}_I} = m$, we get the probability for decay,

$$P = \frac{|\mathcal{A}_{fi}|^2}{2mV}(2\pi)^4\delta^{(4)}(p_I - p_F)VT \prod_{\text{final states}} \frac{1}{2E_{\vec{p}_i}V} \tag{87}$$

where, as in the second derivation of Fermi's Golden Rule, we have exchanged one of the delta-functions for the volume of spacetime: $(2\pi)^4\delta^{(4)}(0) = VT$. The amplitudes $\mathcal{A}_{fi}$ are, of course, exactly what we have been computing. (For example, in Equation 30, we saw that $\mathcal{A} = -g$ for a single meson decaying into two nucleons).

We can now divide by T to get the transition function per unit time. But we still have to worry about summing over all final states. There are two steps: the first is to integrate over all possible momenta of the final particles: $V\int d^3p_i/(2\pi)^3$. The factors of spatial volume V in this measure cancel those in Equation 87, while the factors of $1/2E_{\vec{p}_i}$ in Equation 87 conspire to produce the Lorentz invariant measure for 3-momentum integrals. The result is an expression for the density of final states given by the Lorentz invariant measure

$$d\prod = (2\pi)^4\delta^{(4)}(p_F - p_I) \prod_{\text{final states}} \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_{\vec{p}_i}}. \tag{88}$$

The second step is to sum over all final states with different numbers (and possibly types) of particles. This gives us our final expression for the decay probability per unit time, $\Gamma = \dot{P}$,

$$\Gamma = \frac{1}{2m}\sum_{\text{final states}} \int |\mathcal{A}_{fi}|^2 d\prod. \tag{89}$$

Γ is called *the width of the particle.* It is equal to the reciprocal of the half-life, $\tau = 1/\Gamma$.

7.3. Cross Sections

The standard apparatus is a collision of two beams of particles. Sometimes the particles will hit and bounce off each other; sometimes they will pass right through. The fraction of the time that they collide is called the **cross section** and is denoted by σ. If the incoming flux F is defined to be the number of incoming particles per area per unit time, then the total number of scattering events N per unit time is given by

$$N = F\sigma. \tag{90}$$

We would like to calculate σ from quantum field theory. In fact, we can calculate a more sensitive quantity, $d\sigma$ known as the **differential cross section** which is the probability for a given scattering process to occur in the solid angle (θ, ϕ). More precisely

$$d\sigma = \frac{\text{Differential Probability}}{\text{Unit Time } \times \text{ Unit Flux}} = \frac{1}{4E_1E_2V}\frac{1}{F}|\mathcal{A}_{fi}|^2 d\Pi, \tag{91}$$

where we have used the expression for probability per unit time that we computed in the previous subsection. E_1 and E_2 are the energies of the incoming particles.

We now need an expression for the unit flux. For simplicity, let's sit in the center of mass frame of the collision. We have been considering just a single particle per spatial volume V, meaning that the flux is given in terms of the 3-velocities $\vec{v}_i$ as $F = |\vec{v}_1 - \vec{v}_2|/V$. This then gives

$$d\sigma = \frac{1}{4E_1E_2}\frac{1}{|\vec{v}_1 - \vec{v}_2|}|\mathcal{A}_{fi}|^2 d\Pi. \tag{92}$$

This can also be written in terms of the momentum; simply recall that the 3-velocities $\vec{v}_i$ are related to the momenta by $\vec{v} = \vec{p}/m\sqrt{1-v^2} = \vec{p}/p^0$.

Equation 92 is our final expression relating the S-matrix to the differential cross section. You may now take your favorite scattering amplitude, and compute the probability for particles to fly out at your favorite angles. This will involve doing the integral over the phase space of final states, with measure $d\Pi$. Notice that different scattering amplitudes have different momentum dependence and will result in different angular dependence in scattering amplitudes. For example, in ϕ^4 theory the amplitude for tree level scattering was simply $\mathcal{A} = -\lambda$. This results in isotropic scattering. In contrast, for nucleon-nucleon scattering we have schematically $\mathcal{A} \sim (t - m^2)^{-1} + (u - m^2)^{-1}$. This gives rise to angular dependence in the differential cross-section, which follows from the fact that, for example, $t = -2|\vec{p}|^2(1 - \cos\theta)$, where θ is the angle between the incoming and outgoing particles.

8. Green's Functions

So far we have learn to compute scattering amplitudes. These are directly related to cross-sections and decay rates which are physical quantities. However, QFT is relevant for many other questions as well !. For example, we might want to compute the viscosity of the quark gluon plasma, or the optical conductivity in a tentative model of strange metals, or figure out the non-Gaussianity of density perturbations arising in the CMB from novel models of inflation. All of these questions are answered in the framework of quantum field theory by computing elementary objects known as **correlation functions**. In this section we will briefly define correlation functions, explain how to compute them using Feynman diagrams, and then relate them back to scattering amplitudes.

We will denote the **true vacuum of the interacting theory** as $|\Omega\rangle$. We will normalize H such that

$$H|\Omega\rangle = 0, \tag{93}$$

and $\langle\Omega|\Omega\rangle = 1$. Note that this is different from the state we have called $|0\rangle$ which is the **vacuum of the free theory** and satisfies $H_0|0\rangle = 0$. Define

$$G^{(n)}(x_1, ..., x_n) = \langle\Omega|T\phi_H(x_1)...\phi_H(x_n)|\Omega\rangle, \tag{94}$$

where ϕ_H is simply ϕ in the Heisenberg picture of the full theory, rather than the interaction picture that we have been dealing with so far. The $G^{(n)}$ are called *correlation functions*, or **Green's functions**.

There are a number of different ways of looking at these objects which tie together nicely. Let's start by asking how to compute $G^{(n)}$ using Feynman diagrams. We prove the following result

Claim. We use the notation $\phi_1 = \phi(x_1)$, and write ϕ_{1H} to denote the field in the Heisenberg picture, and ϕ_{1I} to denote the field in the interaction picture. Then

$$G^{(n)}(x_1, ..., x_n) = \langle\Omega|T\phi_{1H}...\phi_{nH}|\Omega\rangle = \frac{\langle 0|T\phi_{1I}...\phi_{nI}S|0\rangle}{\langle 0|S|0\rangle}, \tag{95}$$

where the operators of the right hand side are evaluated on $|0\rangle$, the vacuum of the free theory.

Proof. Take $t_1 > t_2 > ... > t_n$. Then we can drop the T operator, and write the numerator of the right hand side as

$$\begin{aligned}\langle 0|U_I(+\infty, t_1)\phi_{1I}U(t1, t2)\phi_{2I}... \quad & \phi_{nI}U_I(t_n, -\infty)|0\rangle \\ & = \langle 0|U_I(+\infty, t_1)\phi_{1H}...\phi_{nH}U_I(t_n, -\infty)|0\rangle,\end{aligned}$$

where we have used all the intermediate factors of $U_I(t_k, t_{k+1}) = T\exp(-i\int_{t_k}^{t_{k+1}} H_I)$ to convert ϕ_I into ϕ_H. Now, let's deal with the two remaining $U(t, \pm\infty)$ at either end of the string of operators. Consider an arbitrary state $|\Psi\rangle$,

$$\langle\Psi|U_I(t,-\infty)|0\rangle = \langle\Psi|U(t,-\infty)|0\rangle, \tag{96}$$

where $U(t,-\infty)$ is the Schrödinger evolution operator, and the equality above follows because $H_0|0\rangle = 0$.

Insert now a complete set of states, which we take to be energy eigenstates of $H = H_0 + H_{\rm int}$,

$$\begin{aligned} \langle\Psi|U(t,-\infty)|0\rangle &= \langle\Psi|U(t,-\infty)\left[|\Omega\rangle\langle\Omega| + \textstyle\sum_{n\neq 0}|n\rangle\langle n|\right]|0\rangle \\ &= \langle\Psi|\Omega\rangle\langle\Omega|0\rangle + \lim_{t'\to-\infty}\textstyle\sum_{n\neq 0} e^{iE_n(t'-t)}\langle\Psi|n\rangle\langle n|0\rangle. \end{aligned} \tag{97}$$

The last term vanishes; This follows from the Riemann-Lebesgue lemma which says that for any well-behaved function

$$\lim_{\mu\to\infty}\int_a^b dx f(x) e^{i\mu x} = 0. \tag{98}$$

Why is this relevant? The point is that the $\sum_n$ in Equation 97 is really an integral, $\int dn$, because all states are part of a continuum due to the momentum. (There is a caveat here: we want the vacuum $|\Omega\rangle$ to be special, so that it sits on its own, away from the continuum of the integral. This means that we must be working in a theory with a mass gap - i.e. with no massless particles). So the Riemann-Lebesgue lemma gives us

$$\lim_{t'\to-\infty}\langle\Psi|U(t,t')|0\rangle = \langle\Psi|\Omega\rangle\langle\Omega|0\rangle \tag{99}$$

(Notice that to derive this result, Peskin and Schroeder instead send $t \to -\infty$ in a slightly imaginary direction, which also does the job).

We now apply the formula 99, to the top and bottom of the right-hand side of Equation 95, using the definition of S from equation 26 to find

$$\frac{\langle 0|\Omega\rangle\langle\Omega|T\phi_{1H}...\phi_{nH}|\Omega\rangle\langle\Omega|0\rangle}{\langle 0|\Omega\rangle\langle\Omega|\Omega\rangle\langle\Omega|0\rangle}, \tag{100}$$

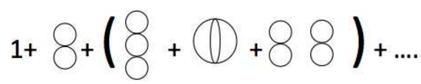
which, using the normalization $\langle\Omega|\Omega\rangle = 1$, gives the left hand side of Equation 95 and completes the proof.

8.1. Connected Diagrams and Vacuum Bubbles

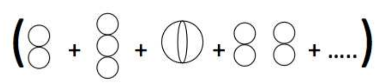
We are getting closer to our goal of computing the Green's functions $G^{(n)}$: we can compute both $\langle 0|T\phi_I(x_1)...\phi_I(x_n)S|0\rangle$ and $\langle 0|S|0\rangle$ using the same methods we developed for

S-matrix elements - namely Dyson's formula and Wick's theorem or, alternatively, Feynman diagrams.

But what about dividing one by the other? What's that all about? In fact, it has a simple interpretation. For the following discussion, we will work in ϕ^4 theory. Since there is no ambiguity in the different types of line in Feynman diagrams, we will represent the ϕ particles as solid lines, rather than the dashed lines that we used previously. Then we have the diagrammatic expansion for $\langle 0|S|0\rangle$.

$$\langle 0|S|0\rangle = $$  (101)

These diagrams are called **vacuum bubbles**. Note that there are exactly 4 lines connected by each vortex. The combinatoric factors (as well as the symmetry factors) associated with each diagram are such that the whole series sums to an exponential,

$$\langle 0|S|0\rangle = \exp$$  (102)

So the amplitude for the vacuum of the free theory to evolve into itself is $\langle 0|S|0\rangle = \exp$ (all distinct vacuum bubbles). A similar combinatoric simplification occurs for generic correlation functions. Remarkably, the vacuum diagrams all add up to give the same exponential. With a little thought one can show that

$$\langle 0|T\phi_1 ... \phi_n|0\rangle = \left(\sum \text{ connected diagrams }\right) \langle 0|S|0\rangle \tag{103}$$

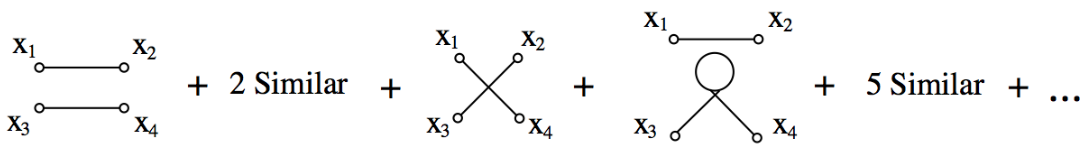
here "connected" means that every part of the diagram is connected to at least one of the external legs.

The upshot of all this is that dividing by $\langle 0|S|0\rangle$ has a very nice interpretation in terms of Feynman diagrams: we need only consider the connected Feynman diagrams, and don't have to worry about the vacuum bubbles. Combining this with Equation 95, we learn that the Green's functions $G^{(n)}(x_1 ... x_n)$ can be calculated by summing over all connected Feynman diagrams,

$$\langle \Omega|T\phi_H(x_1)...\phi_H(x_n)|\Omega\rangle = \sum \textit{Connected Feynman Graphs} \tag{104}$$

An Example: The Four-Point Correlator: $\langle \Omega|T\phi_H(x_1)...\phi_H(x_4)|\Omega\rangle$

As a simple example, let's look at the four-point correlation function in ϕ^4 theory. The sum of connected Feynman diagrams is given by,



All of these are connected diagrams, even though they don't look that connected! The point is that a connected diagram is defined by the requirement that every line is joined to an external leg. An example of a diagram that is not connected is shown in Figure 10. As we have seen, such diagrams are taken care of in shifting the vacuum from $|0\rangle$ to $|\Omega\rangle$.

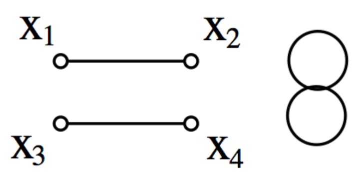


Fig. 10.— Example of a disconnected Feynman diagram

Feynman Rules

The Feynman diagrams that we need to calculate for the Green's functions depend on $x_1, ..., x_n$. This is rather different than the Feynman diagrams that we calculated for the S-matrix elements, where we were working primarily with momentum eigenstates, and ended up integrating over all of space. However, it is rather simple to adapt the Feynman rules that we had earlier in momentum space to compute $G^{(n)}(x1, ..., x_n)$. For ϕ^4 theory, we have

- Draw n external points, $x_1, ..., x_n$, connected by the usual propagators and vertices. Assign a spacetime position y to the end of each line.
- For each line $x - y$ from x to y, write down a factor of the Feynman propagator, $\Delta_F(x - y)$.
- For each vertex at position y, write down a factor of $-i\lambda \int d^4y$.

8.2. From Green's Functions to S-Matrices

Having described how to compute correlation functions using Feynman diagrams, let's now relate them back to the S-matrix elements that we already calculated. The first step is to perform the Fourier transform,

$$\tilde{G}^{(n)}(p_1, ..., p_n) = \int \left[\prod_{i=1}^{n} d^4x_i e^{-ip_i \cdot x_i} \right] G^{(n)}(x_1, ..., x_n). \tag{105}$$

These are very closely related to the S-matrix elements that we have computed above. The difference is that the Feynman rules for $G^{(n)}(x_1, ..., x_n)$, effectively include propagators Δ_F for the external legs, as well as the internal legs. A related fact is that the 4-momenta assigned to the external legs is arbitrary: they are not on-shell. Both of these problems are easily remedied to allow us to return to the S-matrix elements: we need to simply cancel off the propagators on the external legs, and place their momentum back on shell. We have

$$\begin{aligned}\langle p'_1, ..., p'_{n'} | S - 1 | p_1, ..., p_n \rangle = (-i)^{n+n'} \quad & \textstyle\prod_{i=1}^{n'} (p'^2_i - m^2) \prod_{j=1}^{n} (p_j^2 - m^2) \\ & \times \tilde{G}^{(n+n')}(-p'_1, ..., -p'_{n'}, p_1, ..., p_n)\end{aligned} \tag{106}$$

Each of the factors $(p^2 - m^2)$ vanishes once the momenta are placed on-shell. This means that we only get a non-zero answer for diagrams contributing to $G^{(n)}(x_1, ..., x_n)$ which have propagators for each external leg. You might think they all do, but it's not true! Only diagrams that are fully connected, meaning each external point is connected to each other external point, have this property. For example, of the diagrams that we wrote down which contribute to the four-point function $\langle \Omega | T \phi_H(x_1) ... \phi_H(x_4) | \Omega$, only the diagram in Figure 11 will survive the multiplication by on-shell propagators in Equation 106 to contribute to the S-matrix for meson scattering in ϕ^4 theory.

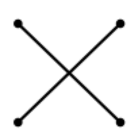


Fig. 11.— The only surviving term in ϕ^4 theory

So what's the point of all of this? We have understood that ignoring the connected diagrams is related to shifting to the true vacuum $|\Omega\rangle$. But other than that, introducing the Green's functions seems like a lot of bother for little reward. The important point is that this provides a framework in which to deal with the true particle states in the interacting theory through renormalization. Indeed, the formula 106, suitably interpreted, remains true even in the interacting theory, taking into account the swarm of virtual particles surrounding asymptotic states. This is the correct way to consider scattering. In this context, Equation 106 is known as the LSZ reduction formula.

REFERENCES

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