

The path to General Relativity

Asaf Pe'er¹

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1. Special relativity

In 1905, Einstein introduced the **special theory of relativity**. This theory was motivated by the **Michelson-Morley experiment**: this fundamental experiment proved that the speed of light, c is the same at any inertial reference frame. Thus, Galilean transformation of velocities between two frames, \mathcal{O} and \mathcal{O}' , $\vec{u}' = \vec{u} + \vec{v}$ (where \vec{v} is the relative velocity of the two frames) fails. Instead, it had to be replaced by **Lorentz transformation**.

The special theory of relativity has 3 great successes:

- First, it conserves the *intervals*, $\delta\tau^2 = \delta t^2 - (\delta x/c)^2$, thereby it is consistent with the results of the Michelson-Morley experiment.
- Second, Galilean transformation (and Newton's laws), familiar from everyday life are obtained as a limiting case in the limit $v \ll c$. Thus, this theory can be viewed as an extension of the Newtonian theory known from the 17th century.
- Third, it is consistent with Maxwell's theory of electromagnetism. Maxwell's equations are invariant under Lorentz transformation, and they admit a wave solution, traveling at the speed of light.

However, the special theory of relativity fails to explain two issues: First, it does not explain the nature of **inertial frames**. Second, it does not explain **gravity**: According to Newton's view, the gravitational force between two bodies, Gm_1m_2/r^2 acts **simultaneously** on both bodies; this is in contradiction with the idea that no signal can move faster than light.

Following the invention of special relativity, Einstein tried for several years to invent a Lorentz invariant theory of gravity, without success. His breakthrough eventually came when he replaced the flat Minkowski space time with a curved space-time, thereby providing a geometrical interpretation to gravity.

¹Physics Dep., University College Cork

2. The principle of relativity and Mach's principle

In Newtonian mechanics there is a unique set of reference frames, known as **inertial reference frames**, within which Newton's first law of motion hold (“if no net force acts on a body it will move in straight line at constant velocity, or will stay at rest if initially at rest”). This is opposed to non-inertial frames (e.g., rotating frames), in which Newton's laws take a different form: in non-inertial frames, fictitious forces, that arise not from any physical interaction but from the acceleration of the non-inertial frame itself exist (e.g., the centrifugal force).

Similarly, inertial frames are particularly important in special relativity, due to the principle of relativity which states that **identical experiments carried in different inertial frames give identical results**.

A question arises: what determines which reference frames are inertial frames ? Newton's answer was that there must exist an *absolute space*, and that the inertial frames are those at rest (or in uniform motion) with respect to that absolute space.

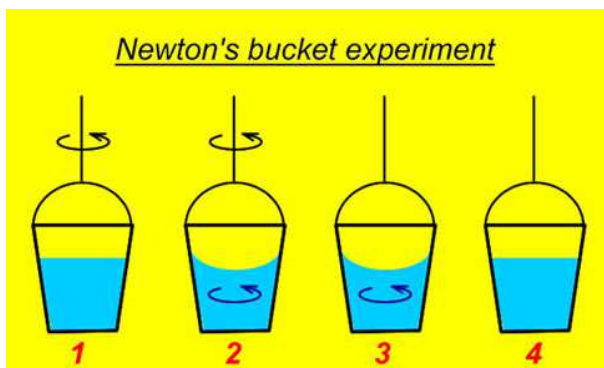


Fig. 1.— Newton's bucket experiment. (1) The bucket rotates, but the water remain still. (2) After a while, the water rotates as well, their surface gets a concave shape. (3) The bucket stops rotating, the water are still rotating and their surface is still concave. (4) The water are at rest. Newton considered this as a prove of the existence of absolute space.

In order to demonstrate his argument, Newton gave the known “rotating bucket” example (see Figure 1): consider a bucket full of water, hanged by a cord. If the cord is twisted up tightly on itself and then the bucket is released, it begins to spin rapidly, not only with respect to the experimenter, but also in relation to the water it contains. However, at this point, the water are not rotating, and their surface is flat. Eventually, the surface of the water assumes a concave shape, as they acquire the motion of the spinning bucket. This concave shape shows that the water is rotating, although at this stage they are at rest with

respect to the bucket. Thus, the concave shape of the water imply that they rotate *relative to something else*, which Newton’s interpreted as absolute space. The motion is thus *absolute*, not relative.

However, *Ernst Mach* had a different interpretation of these results: “*Newton’s experiment with the rotating vessel of water simply informs us that the relative rotation of the water with respect to the sides of the vessel produces no noticeable centrifugal forces, but that such forces are produced by its relative rotations with respect to the mass of the earth and other celestial bodies.*”

Thus, Mach hypothesis was that there is some influence of “the mass of earth and other celestial bodies” which determines the inertial frame. This is known as **Mach’s principle**. In other words: Mach claimed that it is the interaction with the mass of the environment (e.g., galaxies) that determines the inertia. Inertia is thus determined relative to the large scale mass distribution of the universe, rather than relative to some “absolute” space, as claimed by Newton.

This is thus a conceptual difference: Newton believed that inertial forces arise as a result of acceleration with respect to an absolute space, while Mach argued that these forces are caused by acceleration with respect to the mass of celestial bodies. Interestingly, Einstein’s solution took part of both approaches: Einstein showed that inertial frames are determined by local gravitational field (similar to Mach’s approach). However, once in an inertial frame, the laws of motion are completely unaffected by the presence of nearby masses (similar to Newton’s idea).

3. Equivalence of inertial and gravitational mass and the equivalence principle

Already Newton noted that the *inertial mass* that appears in his second law of motion, $\vec{F} = m_I \vec{a}$ is **not necessarily equivalent** to the *gravitational mass* that appears in the laws of gravity, $\vec{F} = -GMm_g \hat{r}/r^2$ (can also be written as $\vec{F} = m_G \vec{g}$).

If the ratio $m_I/m_g \neq 1$ for different bodies, it would imply different acceleration for different bodies in a gravitational field. Already Newton tested whether $m_I = m_G$. Notable experiments by **Eötvös** showed that

$$m_I = m_G, \tag{1}$$

to a great accuracy, better than 10^{-11} (with modern day technology); see Figure 2.

The equivalence between inertial mass and gravitational mass led Einstein to suggest the **principle of equivalence**, which states that

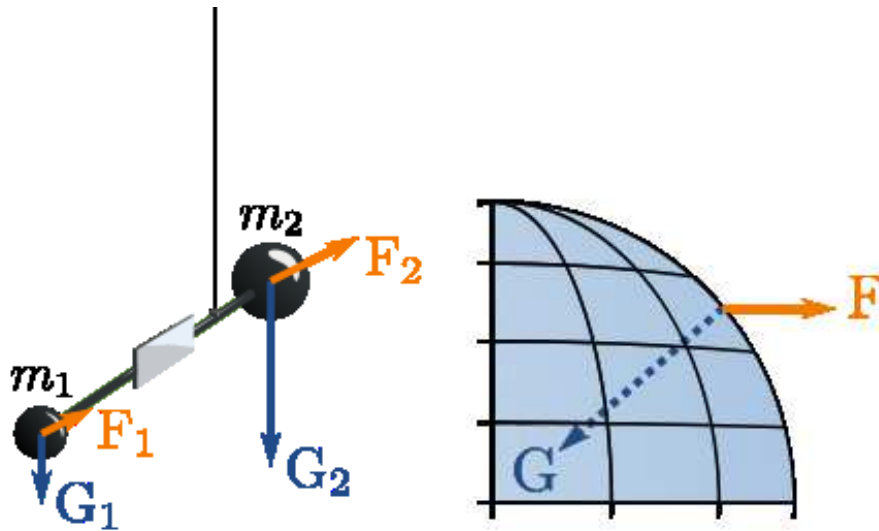


Fig. 2.— Schematic view of Eötvös experiment (left). Two weights (of different materials) are hanged at the end of a beam suspended by a wire at the center. The masses are in equilibrium with the forces of gravity, centripetal acceleration due to the earth’s rotation and the tension in the wire. If $m_I/m_G \neq 1$, the ratio F_1/F_2 would differ than the ratio G_1/G_2 , and the rod would rotate. This rotation would be detected by a beam of light reflected from the mirror. Right: The direction of the centrifugal force in relation to gravity in Budapest latitude, where Eötvös carried his experiment.

At every point in arbitrary gravitational field, it is possible to choose a “locally inertial” coordinate system, such that (within a sufficiently small region of that point) the laws of nature take the same form as in unaccelerated Cartesian coordinate system.

The equivalence principle implies that free-fall and inertial motion are physically equivalent. Thus, an observer in a free-falling lift has no experimental way to determine the gravity of earth. Alternatively, an observer in a windowless room cannot distinguish between being on the surface of the Earth and being in a spaceship in deep space accelerating at $1g$ (to first order).

Using this principle, one can determine the effect of gravitation on arbitrary physical system, but it does not determine the equations for gravity itself. Essentially, Einstein realized that the solution is to identify gravitational field with the component of the metric

tensor of Riemannian space-time geometry. The principle of equivalence is incorporated into this formalism through the requirement that the physical equations are invariant under *general* coordinate transformations, not only Lorentz transformations.

4. Riemannian geometry

The Greek mathematician *Euclid* showed that geometry can be deduced from a few definitions, axioms and postulates. In particular, Euclid’s 5th postulate (the parallel postulate) states that: “if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.”

This can be shown to be equivalent to the statement that “the sum of angles in a triangle is 180° ” (Legendre). However, by the late 18th century - early 19th century, it became clear that one can think of geometries which are very similar to Euclidean geometry in many aspects, but do not fulfill Euclid’s 5th postulate. An example of such a geometry is a triangle drawn on a sphere, see Figure 3. These geometries are known as **non-Euclidean geometries**. The sphere example is also called *two dimensional space with constant positive curvature*. In fact, when we discuss cosmology, we will see that the space in which we actually live in may be a three-dimensional space of constant curvature.

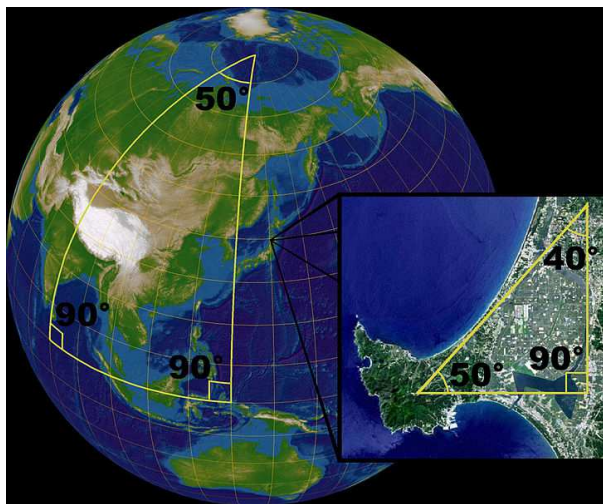


Fig. 3.— On a sphere, the sum of angles of a triangle is always larger than 180° . This is because the surface of a sphere is not a Euclidean space. However, *locally* the laws of Euclidean geometry are good approximation.

It was the great mathematician *Carl Friedrich Gauss* who studied the properties on non-Euclidean spaces, that discriminated between the *inner* properties of a surface, and *outer* properties. The *inner* properties are the properties as measured by a tiny bug living on that surface, while the *outer* properties are the properties embedded in a higher-dimensional space (e.g., the surface of the earth is 2-dimensional embedded in a 3-d space).

Gauss realized that an essential *inner* property of any surface is a function called the **metric**, $d = d(x, y)$, which gives the distance between points x and y *along the shortest possible path on the surface*. For example a *cone* or a *cylinder* have the same inner properties as a plane, since a plane can be rolled into a cone or a cylinder without stretching or tearing. However, a *sphere* cannot be unrolled into a plane surface without distortion - as any cartographer know !. Therefore, its local inner properties are *not the same* as the plane's. Gauss was able to show that the metric function (and its derivatives) are sufficient to determine the inner properties of the space.

It was Gauss's student, *Bernhard Riemann* that further generalized the concepts of non-Euclidean geometry. Riemannian geometry generalizes Euclidean geometry to spaces that are not necessarily flat, although they still resemble the Euclidean space at each point infinitesimally, i.e. in the first order of approximation. Various concepts based on length, such as the arc length of curves, area of plane regions, and volume of solids all possess natural analogies in Riemannian geometry. The notion of a directional derivative of a function from multi-variable calculus is extended in Riemannian geometry to the notion of a covariant derivative of a tensor.

5. General relativity

Using the equivalent principle, Einstein was able to show that the addition of a gravitational field to an otherwise empty space can be viewed as modifying the inner properties of an otherwise flat space-time, making it curved in a similar way to a Riemannian space-time geometry.

In this course, we begin by reviewing physics in a flat space-time: the Minkowski space, known from special relativity. The construction, though, will be done in a way that will enable to generalize the equations to curved space time. We will then study the properties of curved space-time, and show how the laws of physics are written in such a curved space-time.

We will later see how gravity act to curve the space time, by deriving the Einstein's field equation. We will discuss one particular solution to Einstein's equation - that is

Schwarzschild metric, which describes singularity (also known as black hole). We will discuss the paths of particles in such a geometry.

In the final part of this course, we will discuss the structure and evolution of the universe on the largest scale - a branch of science known as **cosmology**. We will discuss the appropriate metric that describes the universe, known as **Friedmann-Robertson-Walker (FRW) metric**, and its observational consequences.