Partial derivatives

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1. Basic definition

Consider f to be a function of a single variable, x. We thus write $f = f(x)$, namely for any value of the independent variable $(x = x_0)$, $f(x = x_0)$ returns a value. In this sense, the function f can be considered as a "box", which gets an input (x_0) and returned an output $(f(x_0))$.

The derivative of a function $f = f(x)$ at $x = x_0$ is defined by

$$\frac{df(x_0)}{dx} \equiv \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \tag{1}$$

Consider now a function f of **two** independent variables, say x and y: $f = f(x, y)$. An example is the depth of a river, or the height of a mountain: at any point inside the river, the depth is different. In this case, one can define a **partial derivative** of f with respect to **one** of the independent variables, say x, at the point (x_0, y_0) as follows:

$$\frac{\partial f(x_0, y_0)}{\partial x} \equiv \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}. \tag{2}$$

A similar definition would hold for the partial derivative with respect to y, $\partial f(x_0, y_0)/\partial y$.

Example: assume $f(x, y) = x^3 + 4x^2 y$. Then

$$\frac{\partial f}{\partial x} = 3x^2 + 8xy; \qquad \frac{\partial f}{\partial y} = 4x^2. \tag{3}$$

Clearly, $\partial f/\partial x \neq \partial f/\partial y$.

The **total differential** of a function $f = f(x, y)$ at point (x_0, y_0) is defined by

$$\Delta f(x_0, y_0) \equiv \frac{\partial f(x_0, y_0)}{\partial x} \Delta x + \frac{\partial f(x_0, y_0)}{\partial y} \Delta y. \tag{4}$$

The total differential gives the (small) change in the value of f as a result of a simultaneous small changes Δx, Δy in the values of the independent parameters x, y.

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2. Basic physical application

Consider a situation in which both x and y are function of a third variable, say t. In this case one can use the chain rule, to write the *full* derivative of f with respect to t:

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dt}\right). \tag{5}$$

Note that while f is a function of two variables, (x, y), we assumed that x is a function of only one variable (t), and similarly y. Hence we used the partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$, but the full derivatives dx/dt and dy/dt.

In the specific case where $x = t$ (or: y is a function of x), Equation 5 becomes

$$\frac{df}{dx} = \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dx}\right) + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dx}\right) = \frac{\partial f}{\partial x} + \left(\frac{\partial f}{\partial y}\right)\left(\frac{dy}{dx}\right). \tag{6}$$

Equation 6 describes a situation in which a quantity f depends on another quantity (here, x) both directly, and also indirectly, being dependent on another quantity (y), which, by itself depends on x. This situation is, in fact, quiet common in physics.

2.1. The continuity Equation (differential form)

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As a specific example, consider the continuity equation, $dm/dt = Av\rho = Const$. We can take the derivative of this equation, to get

$$\frac{d}{dt}(Av\rho) = 0 \tag{7}$$

Consider a cubic volume element, of side area A, which is held constant. As A is constant, we can take it off the derivative in Equation 7. However, the density may be changing, both with time and x. Similarly, the velocity may be changing with x. We can write Equation 6 in the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) = \frac{\partial f}{\partial t} + \left(\frac{\partial f}{\partial x}\right)v \tag{8}$$

Applying this result in Equation 7, one finds

$$\begin{aligned} \frac{d(v\rho)}{dt} &= \frac{\partial(v\rho)}{\partial t} + \frac{\partial(v\rho)}{\partial x}v \\ &= v\frac{\partial\rho}{\partial t} + \rho\frac{\partial v}{\partial t} + v^2\frac{\partial\rho}{\partial x} + \rho v\frac{\partial v}{\partial x} = 0 \end{aligned} \tag{9}$$

For time-independent velocity, $\partial v/\partial t = 0$. Eliminating the 2nd term in Equation 9 and dividing the other terms by v, one gets

$$\begin{aligned} \frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \rho\frac{\partial v}{\partial x} = 0 \\ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0. \end{aligned} \quad (10)$$

Equation 10 is the differential form of the continuity equation. It gives a further insight: the first term represents the change in time of the density (ρ) in a given fluid element. The second term represents the inflow (or outflow) of material through the boundaries of this fluid element, in the x direction.