Quantization of the Dirac Field

Asaf Pe'er[1]

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This part of the course is based on Refs. [1] and [2].

1. Introduction

After deriving the Dirac Lagrangian:

$$\mathcal{L} = \bar{\psi}(x)(i\partial\!\!\!/ - m)\psi(x), \tag{1}$$

it is now time to quantize it.

We begin by the naive approach, following the path we took when treating scalar fields. Soon enough we will see that it will not work, and we will have to reconsider how to quantize this theory.

2. How Not to Quantize the Dirac Field: a Lesson in Spin and Statistics

We start in the usual way and define the momentum,

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger. \tag{2}$$

Thus, for the Dirac Lagrangian, the momentum conjugate to ψ is $i\psi^\dagger$. It does not involve the time derivative of ψ. This is as it should be for an equation of motion that is first order in time, rather than second order. This is because we need only specify ψ and $\psi^\dagger$ on an initial time slice to determine the full evolution.

To quantize the theory, we proceed in the usual way, and promote the field ψ and its momentum $\psi^\dagger$ to operators, satisfying the canonical commutation relations:

$$\begin{array}{rcl} [\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] & = & \left[{\psi_\alpha}^\dagger(\vec{x}), {\psi_\beta}^\dagger(\vec{y})\right] = 0; \\ \left[\psi_\alpha(\vec{x}), {\psi_\beta}^\dagger(\vec{y})\right] & = & \delta_{\alpha\beta}\delta^{(3)}(\vec{x} - \vec{y}). \end{array} \tag{3}$$

This is the step that we will have to reconsider soon.

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[1]Physics Dep., University College Cork

Since we are dealing with a free theory, where any classical solution is a sum of plane waves, we may write the quantum operators as

$$\begin{array}{l}\psi(\vec{x}) = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}u^s(\vec{p})e^{+i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right]\\ \psi^\dagger(\vec{x}) = \sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^{s\dagger}_{\vec{p}}u^s(\vec{p})^\dagger e^{-i\vec{p}\cdot\vec{x}} + c^s_{\vec{p}}v^s(\vec{p})^\dagger e^{+i\vec{p}\cdot\vec{x}}\right],\end{array} \tag{4}$$

where the operators $b^{s\dagger}_{\vec{p}}$ create particles associated to the spinors $u^s(\vec{p})$, while $c^{s\dagger}_{\vec{p}}$ create particles associated to $v^s(\vec{p})$.

Similar to the scalar fields, the commutation relations of the fields imply commutation relations for the annihilation and creation operators:

$$\begin{array}{l}\left[b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\right] = (2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q})\\ \left[c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\right] = -(2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q}),\end{array} \tag{5}$$

with all other commutators vanishing. Note the minus sign in the $[c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}]$ term. It's not yet obvious that it will cause a problem, but we should be aware of it. For now, let's just carry on.

Let us show this. Similar to what we did in scalar fields, we will work one way, showing that the commutation relations $[b, b^\dagger]$ and $[c, c^\dagger]$ reproduce the fields commutator, equation 3:

$$\begin{aligned}[\psi(\vec{x}), \psi^\dagger(\vec{y})] &= \sum_{r,s}\int \frac{d^3p d^3q}{(2\pi)^6}\frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}}\Big([b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}]u^r(\vec{p})u^s(\vec{q})^\dagger e^{i(\vec{x}\cdot\vec{p}-\vec{y}\cdot\vec{q})}\\ &\qquad\qquad + [c^{r\dagger}_{\vec{p}}, c^s_{\vec{q}}]v^r(\vec{p})v^s(\vec{q})^\dagger e^{-i(\vec{x}\cdot\vec{p}-\vec{y}\cdot\vec{q})}\Big)\\ &= \sum_s \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(u^s(\vec{p})\bar{u}^s(\vec{p})\gamma^0 e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + v^s(\vec{p})\bar{v}^s(\vec{p})\gamma^0 e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right)\end{aligned} \tag{6}$$

We now use the outer product formulae that we derived earlier ("The Dirac Equation", Equations 124, 125: $\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p} + m$ and $\sum_{s=1}^{2} v^s(\vec{p})\bar{v}^s(\vec{p}) = \not{p} - m$), to write

$$\begin{aligned}[\psi(\vec{x}), \psi^\dagger(\vec{y})] &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left((\not{p}+m)\gamma^0 e^{i\vec{p}\cdot(\vec{x}-\vec{y})} + (\not{p}-m)\gamma^0 e^{-i\vec{p}\cdot(\vec{x}-\vec{y})}\right)\\ &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left((p_0\gamma^0 + p_i\gamma^i + m)\gamma^0 + (p_0\gamma^0 - p_i\gamma^i - m)\gamma^0\right)e^{+i\vec{p}\cdot(\vec{x}-\vec{y})}\end{aligned}$$

where, in the second term, we have changed $\vec{p} \to -\vec{p}$ under the integration sign. Using now $p_0 = E_{\vec{p}}$, we have

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \int \frac{d^3p}{(2\pi)^3}e^{+i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta^{(3)}(\vec{x}-\vec{y}), \tag{7}$$

which completes the proof. Note that we used the minus sign in the $[c, c^\dagger]$ term in moving to the second line in Equation 6.

2.1. The Hamiltonian

Let us now construct the Hamiltonian for the theory. Using the conjugate momenta, $\pi = i\psi^\dagger$, we have

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^i\partial_i + m)\psi \tag{8}$$

The Hamiltonian $H = \int d^3x\mathcal{H}$ agrees with the conserved energy computed using Noether's theorem ("The Dirac Equation", Equation 90)

Next we turn the Hamiltonian into an operator. For that, let us look at

$$(-i\gamma^i\partial_i + m)\psi = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[b^s_{\vec{p}}(-\gamma^i p_i + m)u^s(\vec{p})e^{+i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}}(\gamma^i p_i + m)v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right]$$

where we have left the sum over $s = 1, 2$ implicit. There is a small subtlety with the minus signs in deriving this equation that arises from the use of the Minkowski metric in contracting indices, so that $\vec{p}\cdot\vec{x} = \sum_i x^i p^i = -x^i p_i$. We now use the defining Equations for the spinors $u^s(\vec{p})$ and $v^s(\vec{p})$ given in "The Dirac Equation", Equations 101 and 107, to replace

$$(-\gamma^i p_i + m)u^s(\vec{p}) = \gamma^0 p_0 u^s(\vec{p}) \quad \text{and} \quad (\gamma^i p_i + m)v^s(\vec{p}) = -\gamma^0 p_0 v^s(\vec{p}) \tag{9}$$

and write

$$(-i\gamma^i\partial_i + m)\psi = \int \frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\gamma^0\left[b^s_{\vec{p}}u^s(\vec{p})e^{+i\vec{p}\cdot\vec{x}} - c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right]. \tag{10}$$

We can use this to write the operator Hamiltonian as

$$\begin{aligned} H &= \int d^3x\psi^\dagger\gamma^0(-i\gamma^i\partial_i + m)\psi \\ &= \int \tfrac{d^3xd^3pd^3q}{(2\pi)^6}\sqrt{\tfrac{E_{\vec{p}}}{4E_{\vec{q}}}}\left[b^{r\dagger}_{\vec{q}}u^r(\vec{q})^\dagger e^{-i\vec{q}\cdot\vec{x}} + c^r_{\vec{q}}v^r(\vec{q})^\dagger e^{+i\vec{q}\cdot\vec{x}}\right]\cdot \\ &\qquad \left[b^s_{\vec{p}}u^s(\vec{p})e^{+i\vec{p}\cdot\vec{x}} - c^{s\dagger}_{\vec{p}}v^s(\vec{p})e^{-i\vec{p}\cdot\vec{x}}\right] \\ &= \int \tfrac{d^3p}{(2\pi)^3}\tfrac{1}{2}\left[b^{r\dagger}_{\vec{p}}b^s_{\vec{p}}[u^r(\vec{p})^\dagger\cdot u^s(\vec{p})] - c^r_{\vec{p}}c^{s\dagger}_{\vec{p}}[v^r(\vec{p})^\dagger\cdot v^s(\vec{p})]\right. \\ &\qquad \left. - b^{r\dagger}_{\vec{p}}c^{s}_{-\vec{p}}{}^\dagger[u^r(\vec{p})^\dagger\cdot v^s(-\vec{p})] + c^r_{\vec{p}}b^s_{-\vec{p}}[v^r(\vec{p})^\dagger\cdot u^s(-\vec{p})]\right] \end{aligned} \tag{11}$$

where, in the last two terms, we have relabeled $\vec{p}\to-\vec{p}$. We now use the inner product formulae ("The Dirac Equation", Equations 118, 120 and 123):

$$u^{r\dagger}(\vec{p})\cdot u^s(\vec{p}) = v^{r\dagger}(\vec{p})\cdot v^s(\vec{p}) = 2p_0\delta^{rs} \quad \text{and} \quad u^{r\dagger}(\vec{p})\cdot v^s(-\vec{p}) = v^{r\dagger}(\vec{p})\cdot u^s(-\vec{p}) = 0,$$

to write

$$\begin{aligned} H &= \int \tfrac{d^3p}{(2\pi)^3}E_{\vec{p}}\left(b^{s\dagger}_{\vec{p}}b^s_{\vec{p}} - c^s_{\vec{p}}c^{s\dagger}_{\vec{p}}\right) \\ &= \int \tfrac{d^3p}{(2\pi)^3}E_{\vec{p}}\left(b^{s\dagger}_{\vec{p}}b^s_{\vec{p}} - c^{s\dagger}_{\vec{p}}c^s_{\vec{p}} + (2\pi)^3\delta^{(3)}(0)\right) \end{aligned} \tag{12}$$

The $\delta^{(3)}(0)$ term is familiar and easily dealt with by normal ordering. However the $-c^\dagger c$ term is a disaster! The Hamiltonian is not bounded below, meaning that our quantum theory makes no sense. Taken seriously it would tell us that we could tumble to states of lower and lower energy by continually producing $c^\dagger$ particles.

Since the above calculation was a little tricky, you might think that it's possible to rescue the theory to get the minus signs to work out right. You can play around with different things, but you will always find this minus sign cropping up somewhere. And, in fact, it's telling us something important that we missed.

3. Fermionic Quantization

The key piece of physics that we missed is that spin 1/2 particles are **fermions**, meaning that they obey Fermi-Dirac statistics with the quantum state picking up a minus sign upon the interchange of any two particles. This fact is embedded into the structure of relativistic quantum field theory: **the spin-statistics theorem** says that integer spin fields must be quantized as bosons, while half-integer spin fields must be quantized as fermions. Any attempt to do otherwise will lead to an inconsistency, such as the unbounded Hamiltonian we saw in Equation 12.

So how do we go about quantizing a field as a fermion? Recall that when we quantized the scalar field, the resulting particles obeyed bosonic statistics because the creation and annihilation operators satisfied the commutation relations,

$$[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}] = 0 \;\Rightarrow\; a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} |0\rangle \equiv |\vec{p}, \vec{q}\rangle = |\vec{q}, \vec{p}\rangle \tag{13}$$

To have states obeying fermionic statistics, we need **anti-commutation relations**, $\{A, B\} \equiv AB + BA$. Rather than Equation 3, we will ask that the spinor fields satisfy

$$\begin{array}{l} \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} = \{\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} = 0; \\ \{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} = \delta_{\alpha\beta}\delta^{(3)}(\vec{x} - \vec{y}). \end{array} \tag{14}$$

We still have the expansion in Equation 4 of ψ and $\psi^\dagger$ in terms of b, $b^\dagger$, c and $c^\dagger$. However, using the same logic that led to Equation 5 gives now

$$\begin{array}{l} \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})\delta^{rs}, \\ \{c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q})\delta^{rs}. \end{array} \tag{15}$$

with all other **anti-commutators** vanishing,

$$\{b^r_{\vec{p}}, b^s_{\vec{q}}\} = \{c^r_{\vec{p}}, c^s_{\vec{q}}\} = \{b^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} = \{b^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} = ... = 0 \tag{16}$$

The calculation of the Hamiltonian proceeds as before, all the way through to the penultimate line of Equation 12. At that stage, we get

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left(b_{\vec{p}}^{s\dagger} b_{\vec{p}}^{s} - c_{\vec{p}}^{s} c_{\vec{p}}^{s\dagger} \right) \\ &= \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left(b_{\vec{p}}^{s\dagger} b_{\vec{p}}^{s} + c_{\vec{p}}^{s\dagger} c_{\vec{p}}^{s} - (2\pi)^3 \delta^{(3)}(0) \right) \end{aligned} \tag{17}$$

The anti-commutators have saved us from the indignity of an unbounded Hamiltonian!. Note that when normal ordering the Hamiltonian we now throw away a negative contribution $-(2\pi)^3\delta^{(3)}(0)$. In principle, this could partially cancel the positive contribution from bosonic fields, and possibly help solving the cosmological constant problem.

3.1. The Fermi-Dirac statistics

Similar to the bosonic case, we define the vacuum state $|0\rangle$ to satisfy

$$b_{\vec{p}}^{s}|0\rangle = c_{\vec{p}}^{s}|0\rangle = 0. \tag{18}$$

Although b and c obey anti-commutation relations, the Hamiltonian (Equation 17) has nice commutation relations with them. You can check that

$$\begin{aligned} [H, b_{\vec{p}}^{r}] = -E_{\vec{p}} b_{\vec{p}}^{r} \ \text{ and } \ [H, b_{\vec{p}}^{r\dagger}] = E_{\vec{p}} b_{\vec{p}}^{r\dagger}; \\ \left[H, c_{\vec{p}}^{r}\right] = -E_{\vec{p}} c_{\vec{p}}^{r} \ \text{ and } \ [H, c_{\vec{p}}^{r\dagger}] = E_{\vec{p}} c_{\vec{p}}^{r\dagger}. \end{aligned} \tag{19}$$

This means that we can again construct a tower of energy eigenstates by acting on the vacuum by $b_{\vec{p}}^{r\dagger}$ and $c_{\vec{p}}^{r\dagger}$ to create particles and antiparticles, similar to the bosonic case. For example, we have the one-particle states

$$|\vec{p}, r\rangle = b_{\vec{p}}^{r\dagger}|0\rangle. \tag{20}$$

The two particle states now satisfy

$$|\vec{p}_1, r_1; \vec{p}_2, r_2\rangle \equiv b_{\vec{p}_1}^{r_1\dagger} b_{\vec{p}_2}^{r_2\dagger}|0\rangle = -|\vec{p}_2, r_2; \vec{p}_1, r_1\rangle \tag{21}$$

confirming that the particles do indeed obey Fermi-Dirac statistics. In particular, we have the **Pauli exclusion principle** $|\vec{p}, r; \vec{p}, r\rangle = 0$. Finally, if we wanted to be sure about the spin of the particle, we could act with the angular momentum operator ("Dirac Equation", equation 93) to confirm that a stationary particle $|\vec{p} = 0, r\rangle$ does indeed carry intrinsic angular momentum 1/2 as expected.

4. Dirac's Hole Interpretation

Lets make a small historical detour, with a lot of physical insight. Dirac originally viewed his equation as a relativistic version of the Schrödinger equation, with ψ interpreted as the wavefunction for a single particle with spin. To reinforce this interpretation, he wrote his equation $(i\partial\!\!\!/ - m)\psi = 0$ as

$$i\frac{\partial\psi}{\partial t} = -i\vec{\alpha}\cdot\vec{\nabla}\psi + m\beta\psi \equiv \hat{H}\psi \tag{22}$$

where $\vec{\alpha} = -\gamma^0\vec{\gamma}$ and $\beta = \gamma^0$. The operator $\hat{H}$ is interpreted as the one-particle Hamiltonian.

Note that this viewpoint is very different than the modern viewpoint, where ψ is a classical field that should be quantized. In Dirac's view, the Hamiltonian of the system is $\hat{H}$ defined above, while for us the Hamiltonian is the field operator defined in Equation 17

Let's, however, see where Dirac's viewpoint leads. With the interpretation of ψ as a single-particle wavefunction, the plane-wave solutions to the Dirac equation, $\psi = u(\vec{p})e^{-ip\cdot x}$ and $\psi = v(\vec{p})e^{+ip\cdot x}$ discussed in "Dirac equation" (equations 100, 106) are thought of as energy eigenstates, with

$$\begin{array}{lcl} \psi = u(\vec{p})e^{-ip\cdot x} & \Rightarrow & i\frac{\partial\psi}{\partial t} = E_{\vec{p}}\psi \\ \psi = v(\vec{p})e^{+ip\cdot x} & \Rightarrow & i\frac{\partial\psi}{\partial t} = -E_{\vec{p}}\psi. \end{array} \tag{23}$$

which look like positive and negative energy solutions. The spectrum is once again unbounded below; there are states $v(\vec{p})$ with arbitrary low energy, $-E_{\vec{p}}$.

At first glance this is disastrous, just like the unbounded field theory Hamiltonian (Equation 12). Dirac postulated an ingenious solution to this problem: since the electrons are fermions (a fact which is put in by hand to Dirac's theory) they obey Pauli's exclusion principle. So we could simply stipulate that in the true vacuum of the universe, all the negative energy states are filled. Only the positive energy states are accessible. These filled negative energy states are referred to as the **Dirac sea**. Although you might worry about the infinite negative charge of the vacuum, Dirac argued that only charge differences would be observable (a trick reminiscent of the normal ordering prescription we used for field operators).

Having avoided disaster by floating on an infinite sea comprised of occupied negative energy states, Dirac realized that his theory made a shocking prediction. Suppose that a negative energy state is excited to a positive energy state, leaving behind a hole. The hole would have all the properties of the electron, except it would carry positive charge. While initially Dirac thought it may be a proton (positrons were not discovered yet), Dirac finally concluded that the hole is a new particle: the positron. Moreover, when a positron comes across an electron, the two can annihilate. **Dirac had predicted anti-matter**, one of the greatest achievements of theoretical physics. It took only a couple of years before the positron was discovered experimentally in 1932.

Although Dirac's physical insight led him to the right answer, we now understand that the interpretation of the Dirac spinor as a single-particle wavefunction is not really correct. For example, Dirac's argument for anti-matter relies crucially on the particles being fermions while, as we have seen already in this course, anti-particles exist for both fermions and bosons. What we really learn from Dirac's analysis is that there is no consistent way to interpret the Dirac equation as describing a single particle. It is instead to be thought of as a classical field which has only positive energy solutions because the Hamiltonian ($H = \int d^3x\psi^\dagger\gamma^0(-i\gamma^i\partial_i + m)\psi$) is positive definite. Quantization of this field then gives rise to both particle and anti-particle excitations.

5. Propagators

We proceed in the same path we took when quantizing the scalar field, and we move now to the Heisenberg picture. We define the spinors $\psi(t, \vec{x})$ at every point in spacetime such that they satisfy the operator equation

$$\frac{\partial\psi}{\partial t} = i[H, \psi]. \tag{24}$$

We solve this by the expansion

$$\begin{aligned} \psi(x) &= \textstyle\sum_{s=1}^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b^s_{\vec{p}} u^s(\vec{p}) e^{-ip\cdot x} + c^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{+ip\cdot x}\right] \\ \psi^\dagger(x) &= \textstyle\sum_{s=1}^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[b^{s\dagger}_{\vec{p}} u^s(\vec{p})^\dagger e^{+ip\cdot x} + c^s_{\vec{p}} v^s(\vec{p})^\dagger e^{-ip\cdot x}\right] \end{aligned} \tag{25}$$

Let's now look at the anti-commutators of these fields. We define the fermionic propagator to be

$$iS_{\alpha\beta} = \{\psi_\alpha(x), \bar{\psi}_\beta(y)\} \tag{26}$$

In what follows we will often drop the indices and simply write $iS(x-y) = \{\psi(x), \bar{\psi}(y)\}$, though remember that $S(x-y)$ is a 4×4 matrix.

Inserting the expansion (Equation 25), we have

$$\begin{aligned} iS(x-y) &= \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{4E_{\vec{p}}E_{\vec{q}}}} \left[\{b^s_{\vec{p}}, b^{r\dagger}_{\vec{q}}\} u^s(\vec{p})\bar{u}^r(\vec{q}) e^{-i(p\cdot x - q\cdot y)} \right. \\ &\qquad \left. + \{c^{s\dagger}_{\vec{p}}, c^r_{\vec{q}}\} v^s(\vec{p})\bar{v}^r(\vec{q}) e^{+i(p\cdot x - q\cdot y)}\right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[u^s(\vec{p})\bar{u}^s(\vec{p}) e^{-ip\cdot(x-y)} + v^s(\vec{p})\bar{v}^s(\vec{p}) e^{+ip\cdot(x-y)}\right] \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left[(\not{p} + m) e^{-ip\cdot(x-y)} + (\not{p} - m) e^{+ip\cdot(x-y)}\right] \end{aligned} \tag{27}$$

where we have used the outer product formulae, $\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p}+m$, and $\sum_{s=1}^{2} v^s(\vec{p})\bar{v}^s(\vec{p}) = \not{p} - m$ we derived in "Dirac Equation" in writing the last line.

We can thus write

$$iS(x-y) = (i\not{\partial}_x + m)(D(x-y) - D(y-x)) \tag{28}$$

in terms of the propagator for a real scalar field $D(x-y)$ which, recall from "Free Fields" can be written as

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}. \tag{29}$$

Comments.

[1] We have seen that for spacelike separation, $(x-y)^2 < 0$, we have $D(x-y) - D(y-x) = 0$. When discussing bosons, we claimed that this fact ensures causality, since

$$[\phi(x), \phi(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0 \tag{30}$$

outside the lightcone.

However, for fermions we now have

$$\{\psi_\alpha(x), \psi_\beta(y)\} = 0 \quad \text{for} \quad (x-y)^2 < 0 \tag{31}$$

outside the lightcone. So what about causality now ? The best that we can say is that all our observables are bilinear in fermions (for example the Hamiltonian in Equation 17). These still commute outside the lightcone. The theory thus remains causal as long as fermionic operators are not observable.

Though this may sounds a little weak, remember that no one has ever seen a physical measuring apparatus come back to minus itself when you rotate by 360 degrees.

[2] At least away from the singularities, the propagator satisfies

$$(i\not{\partial}_x - m)S(x-y) = 0, \tag{32}$$

which follows from the fact that $(\not{\partial}_x^{\,2} + m^2)D(x-y) = 0$, using the mass shell condition, $p^2 = m^2$.

6. The Feynman Propagator

By a similar calculation to that above, we can determine the vacuum expectation value,

$$\begin{aligned} \langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle &= \int \tfrac{d^3p}{(2\pi)^3} \tfrac{1}{2E_{\vec{p}}} (\not{p} + m)_{\alpha\beta} e^{-ip\cdot(x-y)} \\ \langle 0|\bar{\psi}_\beta(y)\psi_\alpha(x)|0\rangle &= \int \tfrac{d^3p}{(2\pi)^3} \tfrac{1}{2E_{\vec{p}}} (\not{p} - m)_{\alpha\beta} e^{+ip\cdot(x-y)}. \end{aligned} \tag{33}$$

We can now define the Feynman propagator $S_F(x-y)$, which is again a 4×4 matrix, as the time ordered product,

$$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle \equiv \begin{cases} \langle 0|\psi(x)\bar{\psi}(y)|0\rangle & x^0 > y^0 \\ \langle 0| - \bar{\psi}(y)\psi(x)|0\rangle & y^0 > x^0 \end{cases} \tag{34}$$

Notice the minus sign! It is necessary for Lorentz invariance. When $(x-y)^2 < 0$, there is no invariant way to determine whether $x^0 > y^0$ or $y^0 > x^0$. In this case the minus sign is necessary to make the two definitions agree since $\{\psi(x), \bar{\psi}(y)\} = 0$ outside the lightcone.

We have the 4-momentum integral representation for the Feynman propagator,

$$S_F(x-y) = i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{\gamma\cdot p + m}{p^2 - m^2 + i\epsilon} \tag{35}$$

which satisfies $(i\not{\partial}_x - m)S_F(x-y) = i\delta^{(4)}(x-y)$, so that S_F is a Green's function for the Dirac operator.

The minus sign in Equation 34 also occurs for any string of operators inside a time ordered product $T(...)$. While bosonic operators commute inside T, fermionic operators anti-commute.

We have this same behavior for normal ordered products as well, with fermionic operators obeying $:\psi_1\psi_2: = -:\psi_2\psi_1:$. With the understanding that all fermionic operators anti-commute inside T and $::$, Wick's theorem proceeds just as in the bosonic case. We define the contraction

$$\overbrace{\psi(x)\bar{\psi}(y)} = T(\psi(x)\bar{\psi}(y)) - :\psi(x)\bar{\psi}(y): = S_F(x-y). \tag{36}$$

7. The Yukawa Theory

Let us now return to the Yukawa theory, but now introduce interactions between Dirac fermion of mass m, and a real scalar field of mass μ. The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \lambda\phi\bar{\psi}\psi \tag{37}$$

This is the proper version of the scalar Yukawa theory we encountered earlier. Couplings of this type do appear in the standard model, between fermions and the Higgs boson. In that context, the fermions can be leptons (such as the electron) or quarks.

Yukawa originally proposed an interaction of this type as an effective theory of nuclear forces. With an eye to this, we will again refer to the ϕ particles as mesons, and the ψ particles as nucleons. Except, this time, the nucleons have spin. (This is still not a particularly realistic theory of nucleon interactions, not least because we are omitting isospin. Moreover, in Nature the relevant mesons are pions which are pseudoscalars, so a coupling of the form $\phi\bar{\psi}\gamma^5\psi$ would be more appropriate. We will comment on this shortly later on).

Note the dimensions of the various fields. We still have $[\phi] = 1$, but the kinetic terms require that $[\psi] = 3/2$. Thus, unlike in the case with only scalars, the coupling is dimensionless: $[\lambda] = 0$.

We proceed as we did in the pure bosonic case, firstly computing the amplitude of a particular scattering process then, with that calculation as a guide, writing down the Feynman rules for the theory.

7.1. The Amplitude of Nucleon Scattering

We study (again) $\psi\psi \to \psi\psi$ scattering. This is the same calculation we performed in "Interacting Fields", except now the fermions have spin. Our initial and final states are

$$\begin{aligned} |i\rangle &= \sqrt{4E_{\vec{p}}E_{\vec{q}}}\, b^{s\,\dagger}_{\vec{p}} b^{r\,\dagger}_{\vec{q}}|0\rangle \equiv |\vec{p}, s; \vec{q}, r\rangle \\ |f\rangle &= \sqrt{4E_{\vec{p}\,'}E_{\vec{q}\,'}}\, b^{s'\,\dagger}_{\vec{p}\,'} b^{r'\,\dagger}_{\vec{q}\,'}|0\rangle \equiv |\vec{p}\,', s'; \vec{q}\,', r'\rangle \end{aligned} \tag{38}$$

We need to be a little cautious about minus signs, because the $b^\dagger$'s now anti-commute. In particular, we should be careful when we take the adjoint. We have

$$\langle f| = \sqrt{4E_{\vec{p}\,'}E_{\vec{q}\,'}}\,\langle 0| b^{r'}_{\vec{q}\,'} b^{s'}_{\vec{p}\,'} \tag{39}$$

We want to calculate the order λ^2 terms from the S-matrix element $\langle f|S - 1|i\rangle$:

$$\frac{(-i\lambda)^2}{2}\int d^4x_1 d^4x_2\, T\left(\bar{\psi}(x_1)\psi(x_1)\phi(x_1)\bar{\psi}(x_2)\psi(x_2)\phi(x_2)\right) \tag{40}$$

where, as usual, all fields are in the interaction picture. Just as in the bosonic calculation, the contribution to nucleon scattering comes from the contraction

$$:\bar{\psi}(x_1)\psi(x_1)\bar{\psi}(x_2)\psi(x_2):\ \overbrace{\phi(x_1)\phi(x_2)} \tag{41}$$

We do have to be careful about how the spinor indices are contracted. Let's start by looking at how the fermionic operators act on $|i\rangle$. We expand the ψ fields, leaving the $\bar{\psi}$ fields alone for now. We may ignore the $c^\dagger$ pieces in ψ since they give no contribution at order λ^2. We have

$$: \bar{\psi}(x_1)\psi(x_1)\bar{\psi}(x_2)\psi(x_2) : b_{\vec{p}}^{s\dagger} b_{\vec{q}}^{r\dagger}|0\rangle = -\int \frac{d^3k_1 d^3k_2}{(2\pi)^6}[\bar{\psi}(x_1)\cdot u^m(\vec{k}_1)][\bar{\psi}(x_2)\cdot u^n(\vec{k}_2)] \frac{e^{-ik_1\cdot x_1 - ik_2\cdot x_2}}{\sqrt{4E_{\vec{k}_1}E_{\vec{k}_2}}} b_{\vec{k}_1}^m b_{\vec{k}_2}^n b_{\vec{p}}^{s\dagger} b_{\vec{q}}^{r\dagger}|0\rangle \tag{42}$$

where we have used square brackets [ ] to show how the spinor indices are contracted. The minus sign that sits out front came from moving $\psi(x_1)$ past $\bar{\psi}(x_2)$. Now anti-commuting the b's past the $b^\dagger$'s, we get

$$= \frac{-1}{2\sqrt{E_{\vec{k}_1}E_{\vec{k}_2}}}\Big([\bar{\psi}(x_1)\cdot u^r(\vec{q})][\bar{\psi}(x_2)\cdot u^s(\vec{p})]e^{-ip\cdot x_2 - iq\cdot x_1} - [\bar{\psi}(x_1)\cdot u^s(\vec{p})][\bar{\psi}(x_2)\cdot u^r(\vec{q})]e^{-ip\cdot x_1 - iq\cdot x_2}\Big)|0\rangle \tag{43}$$

Note, in particular, the relative minus sign that appears between these two terms.

Now let's see what happens when we hit this with $\langle f|$. Look at

$$\langle 0|b_{\vec{q}'}^{r'\dagger} b_{\vec{p}'}^{s'\dagger}[\bar{\psi}(x_1)\cdot u^r(\vec{q})][\bar{\psi}(x_2)\cdot u^s(\vec{p})]|0\rangle = \frac{e^{+ip'\cdot x_1 + iq'\cdot x_2}}{2\sqrt{E_{\vec{p}'}E_{\vec{q}'}}}[\bar{u}^{s'}(\vec{p}')\cdot u^r(\vec{q})][\bar{u}^{r'}(\vec{q}')\cdot u^s(\vec{p})] - \frac{e^{+ip'\cdot x_2 + iq'\cdot x_1}}{2\sqrt{E_{\vec{p}'}E_{\vec{q}'}}}[\bar{u}^{r'}(\vec{q}')\cdot u^r(\vec{q})][\bar{u}^{s'}(\vec{p}')\cdot u^s(\vec{p})]$$

The $[\bar{\psi}(x_1)\cdot u^s(\vec{p})][\bar{\psi}(x_2)\cdot u^r(\vec{q})]$ term in Equation 43 doubles up with this, canceling the factor of 1/2 in front of Equation 40. Meanwhile, the $1/\sqrt{E}$ terms cancel the relativistic state normalization. Putting everything together, we have the following expression for $\langle f|S-1|i\rangle$,

$$(-i\lambda)^2\int \frac{d^4x_1 d^4x_2 d^4k}{(2\pi)^4}\frac{ie^{ik\cdot(x_1-x_2)}}{k^2-\mu^2+i\epsilon}\Big([\bar{u}^{s'}(\vec{p}')\cdot u^s(\vec{p})][\bar{u}^{r'}(\vec{q}')\cdot u^r(\vec{q})]e^{+ix_1\cdot(q'-q)+ix_2\cdot(p'-p)} - [\bar{u}^{s'}(\vec{p}')\cdot u^r(\vec{q})][\bar{u}^{r'}(\vec{q}')\cdot u^s(\vec{p})]e^{+ix_1\cdot(p'-q)+ix_2\cdot(q'-p)}\Big)$$

where we have put the ϕ propagator back in. Performing the integrals over x_1 and x_2, this becomes,

$$\int d^4k\frac{(2\pi)^4 i(-i\lambda)^2}{k^2-\mu^2+i\epsilon}\Big([\bar{u}^{s'}(\vec{p}')\cdot u^s(\vec{p})][\bar{u}^{r'}(\vec{q}')\cdot u^r(\vec{q})]\delta^{(4)}(q'-q+k)\delta^{(4)}(p'-p-k) - [\bar{u}^{s'}(\vec{p}')\cdot u^r(\vec{q})][\bar{u}^{r'}(\vec{q}')\cdot u^s(\vec{p})]\delta^{(4)}(p'-q+k)\delta^{(4)}(q'-p-k)\Big)$$

Finally, writing the S-matrix element in terms of the amplitude in the usual way, $\langle f|S-1|i\rangle = i\mathcal{A}(2\pi)^4\delta^{(4)}(p+q-p'-q')$, we have

$$\mathcal{A} = (-i\lambda)^2\left(\frac{[\bar{u}^{s'}(\vec{p}')\cdot u^s(\vec{p})][\bar{u}^{r'}(\vec{q}')\cdot u^r(\vec{q})]}{(p'-p)^2-\mu^2+i\epsilon} - \frac{[\bar{u}^{s'}(\vec{p}')\cdot u^r(\vec{q})][\bar{u}^{r'}(\vec{q}')\cdot u^s(\vec{p})]}{(q'-p)^2-\mu^2+i\epsilon}\right)$$

which is our final answer for the amplitude.

8. Feynman Rules for Fermions

Its important to bear in mind that the calculation we just did kind of blows. Thankfully the Feynman rules will once again encapsulate the combinatoric complexities and make life easier for us. The rules to compute amplitudes are the following

- To each incoming fermion with momentum p and spin r, we associate a spinor $u^r(\vec{p})$. For outgoing fermions we associate $\bar{u}^r(\vec{p})$. (see Figures 1).

Fig. 1.— Left: an incoming fermion. Right: an outgoing fermion.

- To each incoming anti-fermion with momentum p and spin r, we associate a spinor $\bar{v}^r(\vec{p})$. For outgoing anti-fermions we associate $v^r(\vec{p})$. (See Figures 2).

Fig. 2.— Left: an incoming anti-fermion. Right: an outgoing anti-fermion.

- Each vertex gets a factor of $-i\lambda$.
- Each internal line gets a factor of the relevant propagator.

$$\overset{p\rightarrow}{- - - - - - -} \qquad \frac{i}{p^2-\mu^2+i\epsilon} \text{ for scalars} \tag{44}$$

$$\overset{p\rightarrow}{\longrightarrow\!\!\text{———}} \qquad \frac{i(\not{p}+m)}{p^2-\mu^2+i\epsilon} \text{ for fermions}$$

The arrows on the fermion lines must flow consistently through the diagram (this ensures fermion number conservation). Note that the fermionic propagator is a 4×4

matrix. The matrix indices are contracted at each vertex, either with further propagators, or with external spinors u, $\bar{u}$, v or $\bar{v}$.

- Impose momentum conservation at each vertex, and integrate over undetermined loop momenta.
- Add extra minus signs for statistics. (see Examples below).

8.1. Examples

Let us repeat the same examples as for the scalar Yukawa theory, but now adding the spin.

Nucleon Scattering
For the example we worked out previously, the two lowest order Feynman diagrams are shown in Figure 3.

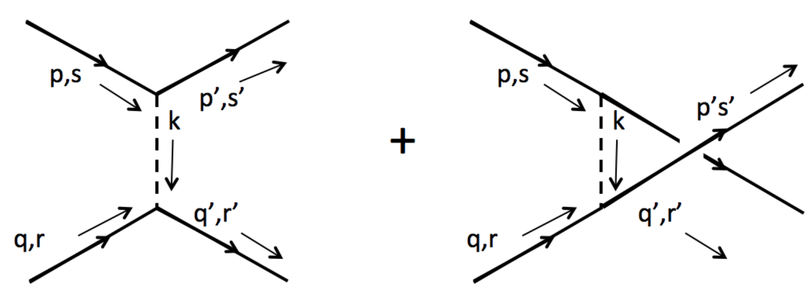


Fig. 3.— The two Feynman diagrams for nucleon scattering.

The second Feynman diagram is drawn with the legs crossed to emphasize the fact that it picks up a minus sign due to statistics. (Note that the way the legs point in the Feynman diagram doesn't tell us the direction in which the particles leave the scattering event: the momentum label does that. The two diagrams above are different because the incoming legs are attached to different outgoing legs). Using the Feynman rules we can read off the amplitude.

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\vec{p}') \cdot u^s(\vec{p})][\bar{u}^{r'}(\vec{q}') \cdot u^r(\vec{q})]}{(p-p')^2 - \mu^2} - \frac{[\bar{u}^{s'}(\vec{p}') \cdot u^r(\vec{q})][\bar{u}^{r'}(\vec{q}') \cdot u^s(\vec{p})]}{(p-q')^2 - \mu^2} \right) \tag{45}$$

The denominators in each term are due to the meson propagator, with the momentum determined by conservation at each vertex. This clearly agrees with the amplitude we computed earlier using Wick's theorem.

Nucleon to Meson Scattering
Let us now look at $\psi\bar{\psi} \to \phi\phi$. The two lowest order Feynman diagrams are shown in Figure 4.

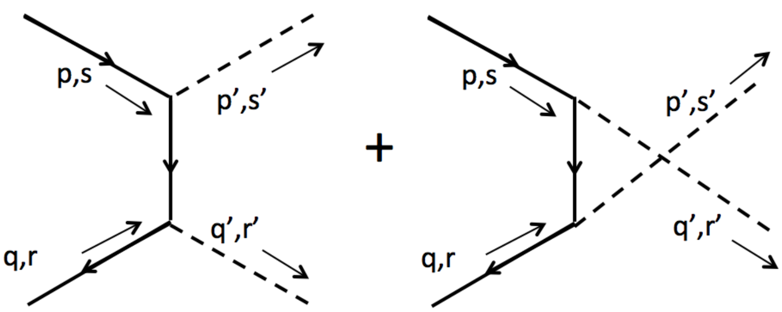


Fig. 4.— The two Feynman diagrams for nucleon to meson scattering.

Applying the Feynman rules, we have

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{\bar{v}^r(\vec{q})[\gamma^\mu(p_\mu - p'_\mu) + m]u^s(\vec{p})}{(p-p')^2 - m^2} + \frac{\bar{v}^r(\vec{q})[\gamma^\mu(p_\mu - q'_\mu) + m]u^s(\vec{p})}{(p-q')^2 - m^2} \right)$$

Since the internal line is now a fermion, the propagator contains factors of $\gamma^\mu(p_\mu - p'_\mu) + m$. This is a 4×4 matrix which sits on the top, sandwiched between the two external spinors. Now the exchange statistics applies to the final meson states. These are bosons and, correspondingly, there is no relative minus sign between the two diagrams.

Nucleon- anti-Nucleon Scattering
For $\psi\bar{\psi} \to \psi\bar{\psi}$, the two lowest order Feynman diagrams are of two distinct types, just like in the bosonic case. They are shown in Figure 5.

The corresponding amplitude is given by,

$$\mathcal{A} = (-i\lambda)^2 \left(-\frac{[\bar{u}^{s'}(\vec{p}') \cdot u^s(\vec{p})][\bar{v}^r(\vec{q}) \cdot v^{r'}(\vec{q}')]}{(p-p')^2 - \mu^2} + \frac{[\bar{v}^r(\vec{q}) \cdot u^s(\vec{p})][\bar{u}^{s'}(\vec{p}') \cdot v^{r'}(\vec{q}')]}{(p+q)^2 - \mu^2 + i\epsilon} \right) \qquad (46)$$

As in the bosonic diagrams, there is again the difference in the momentum dependence in the denominator. But now the difference in the diagrams is also reflected in the spinor contractions in the numerator.

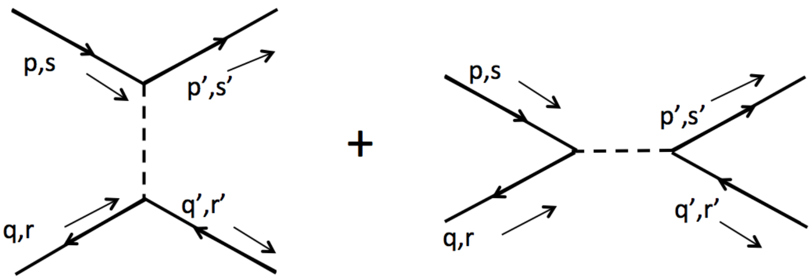


Fig. 5.— The two Feynman diagrams for nucleon - anti-nucleon scattering.

More subtle are the minus signs. The fermionic statistics mean that the first diagram has an extra minus sign relative to the $\psi\psi$ scattering of Figure 3. Since this minus sign will be important when we come to figure out whether the Yukawa force is attractive or repulsive, let's go back to basics and see where it comes from. The initial and final states for this scattering process are

$$\begin{aligned} |i\rangle &= \sqrt{4E_{\vec{p}}E_{\vec{q}}}\, b^{s\,\dagger}_{\vec{p}} c^{r\,\dagger}_{\vec{q}}|0\rangle \equiv |\vec{p},s;\vec{q},r\rangle \\ |f\rangle &= \sqrt{4E_{\vec{p}'}E_{\vec{q}'}}\, b^{s'\,\dagger}_{\vec{p}'} c^{r'\,\dagger}_{\vec{q}'}|0\rangle \equiv |\vec{p}',s';\vec{q}',r'\rangle \end{aligned} \tag{47}$$

The ordering of $b^\dagger$ and $c^\dagger$ in these states **is crucial** and reflects the scattering $\psi\bar{\psi} \to \psi\bar{\psi}$, as opposed to $\psi\bar{\psi} \to \bar{\psi}\psi$ which would differ by a minus sign. The first diagram in Figure 5 comes from the term in the perturbative expansion,

$$\langle f| : \bar{\psi}(x_1)\psi(x_1)\bar{\psi}(x_2)\psi(x_2) : b^{s\,\dagger}_{\vec{p}} c^{r\,\dagger}_{\vec{q}}|0\rangle \sim \langle f|[\bar{v}^m(\vec{k}_1)\cdot\psi(x_1)][\bar{\psi}(x_2)\cdot u^n(\vec{k}_2)]c^m_{\vec{k}_1} b^n_{\vec{k}_2} b^{s\,\dagger}_{\vec{p}} c^{r\,\dagger}_{\vec{q}}|0\rangle$$

where we have neglected a bunch of objects in this equation like $\int d^4k_i$ and exponential factors because we only want to keep track of the minus signs. Moving the annihilation operators past the creation operators, we have

$$+\langle f|[\bar{v}^r(\vec{q})\cdot\psi(x_1)][\bar{\psi}(x_2)\cdot u^s(\vec{p})]|0\rangle \tag{48}$$

Repeating the process by expanding out the $\psi(x_1)$ and $\bar{\psi}(x_2)$ fields, and moving them to the left to annihilate $\langle f|$, we have

$$\langle 0|c^{r'}_{\vec{q}'} b^{s'}_{\vec{p}'} c^{m\,\dagger}_{\vec{l}_1} b^{n\,\dagger}_{\vec{l}_2}[\bar{v}^r(\vec{q})\cdot v^m(\vec{l}_1)][\bar{u}^n(\vec{l}_2)\cdot u^s(\vec{p})]|0\rangle \sim -[\bar{v}^r(\vec{q})\cdot v^{r'}(\vec{q}')][\bar{u}^{s'}(\vec{p}')\cdot u^s(\vec{p})]$$

where the minus sign has appeared from anti-commuting $c^{m\,\dagger}_{\vec{l}_1}$ past $b^{s'}_{\vec{p}'}$. This is the overall minus sign found in Equation 46. One can also follow similar contractions to compute the second diagram in Figure 5.

Meson Scattering

Finally, we can also compute the scattering of $\phi\phi \to \phi\phi$ which, as in the bosonic case, picks up its leading contribution at one-loop.

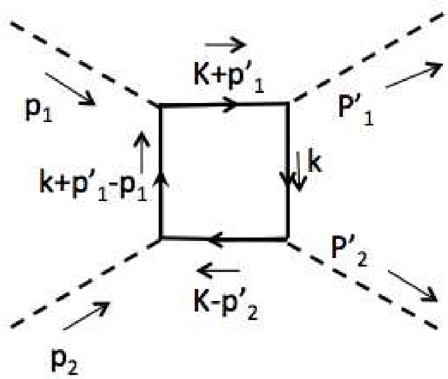


Fig. 6.— Feynman diagram for meson scattering.

The amplitude for the diagram shown in the figure 6 is

$$i\mathcal{A} = -(-i\lambda)^4 \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr} \frac{\not{k}+m}{(k^2-m^2+i\epsilon)} \frac{\not{k}+\not{p}'_1+m}{((k+p'_1)^2-m^2+i\epsilon)} \times \frac{\not{k}+\not{p}'_1-\not{p}_1+m}{((k+p'_1-p_1)^2-m^2+i\epsilon)} \frac{\not{k}-\not{p}'_2+m}{((k-p'_2)^2-m^2+i\epsilon)}$$

Notice that the high momentum limit of the integral is $\int d^4k/k^4$, which is no longer finite, but diverges logarithmically. Again, unfortunately dealing with these integrals are only done in advanced QFT course.

There is an overall minus sign sitting in front of this amplitude. This is a generic feature of diagrams with fermions running in loops: each fermionic loop in a diagram gives rise to an extra minus sign. We can see this rather simply in Figure 7, which involves the expression

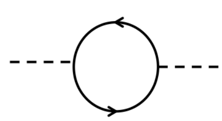


Fig. 7.— Fermionic loop.

$$\begin{aligned}\bar{\psi}_\alpha(x)\,\psi_\alpha(x)\bar{\psi}_\beta(y)\,\psi_\beta(y) &= -\,\psi_\beta(y)\bar{\psi}_\alpha(x)\,\psi_\alpha(x)\bar{\psi}_\beta(y) \\ &= -\mathrm{Tr}(S_F(y-x)S_F(x-y))\end{aligned}$$

After passing the fermion fields through each other, a minus sign appears, sitting in front of the two propagators.

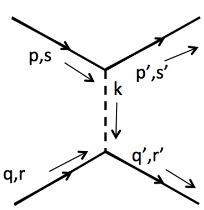
8.2. The Yukawa Potential Revisited

We saw earlier when discussing "Interacting Fields", that the exchange of a real scalar particle gives rise to a universally attractive Yukawa potential between two spin zero particles. Does the same hold for the spin 1/2 particles?

Recall that the strategy to compute the potential is to take the non-relativistic limit of the scattering amplitude, and compare with the analogous result from quantum mechanics. Our new amplitude now also includes the spinor degrees of freedom $u(\vec{p})$ and $v(\vec{p})$. In the non-relativistic limit, $p \to (m, \vec{p})$, and

$$\begin{aligned} u(\vec{p}) &= \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ \sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} \to \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \\ v(\vec{p}) &= \begin{pmatrix} \sqrt{p\cdot\sigma}\xi \\ -\sqrt{p\cdot\bar{\sigma}}\xi \end{pmatrix} \to \sqrt{m}\begin{pmatrix} \xi \\ -\xi \end{pmatrix} \end{aligned} \tag{49}$$

In this limit, the spinor contractions in the amplitude for $\psi\psi \to \psi\psi$ scattering (Equation 45) become $\bar{u}^{s'}\cdot u^s = 2m\delta^{ss'}$, and the amplitude (Figure 3, left) is equal to



$$= -i(-i\lambda)^2(2m)\left(\frac{\delta^{ss'}\delta^{rr'}}{(\vec{p}-\vec{p}')+\mu^2} - \frac{\delta^{s'r}\delta^{r's}}{(\vec{p}-\vec{q}')+\mu^2}\right) \tag{50}$$

The δ symbols tell us that spin is conserved in the non-relativistic limit, while the momentum dependence is the same as in the bosonic case, telling us that once again the particles feel an attractive Yukawa potential,

$$U(\vec{r}) = -\frac{\lambda^2 e^{-\mu r}}{4\pi r} \tag{51}$$

Repeating the calculation for $\psi\bar{\psi} \to \psi\bar{\psi}$, there are two minus signs which cancel each other. The first is the extra overall minus sign in the scattering amplitude (Equation 46), due to the fermionic nature of the particles. The second minus sign comes from the non-relativistic limit of the spinor contraction for anti-particles in Equation 46, which is $\bar{v}^{s'} \cdot v^s = -2m\delta^{ss'}$. These two signs cancel, giving us once again an attractive Yukawa potential in Equation 51.

8.3. Pseudo-Scalar Coupling

Rather than the standard Yukawa coupling, we could instead consider

$$\mathcal{L}_{\rm Yuk} = -\lambda\phi\bar{\psi}\gamma^5\psi \tag{52}$$

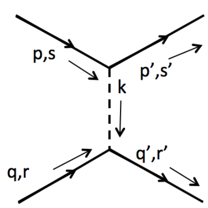
This still preserves parity if ϕ is a pseudoscalar, i.e.

$$P : \phi(t,\vec{x}) \to -\phi(t,-\vec{x}) \tag{53}$$

We can compute in this theory very simply: the Feynman rule for the interaction vertex is now changed to a factor of $-i\lambda\gamma^5$. For example, the Feynman diagrams for $\psi\psi \to \psi\psi$ scattering are again given by Figure 3, with the amplitude now

$$\mathcal{A} = (-i\lambda)^2 \left(\frac{[\bar{u}^{s'}(\vec{p}')\gamma^5 u^s(\vec{p})][\bar{u}^{r'}(\vec{q}')\gamma^5 u^r(\vec{q})]}{(p-p')^2-\mu^2} - \frac{[\bar{u}^{s'}(\vec{p}')\gamma^5 u^r(\vec{q})][\bar{u}^{r'}(\vec{q}')\gamma^5 u^s(\vec{p})]}{(p-q')^2-\mu^2} \right)$$

We could again try to take the non-relativistic limit for this amplitude. But this time, things work a little differently. Using the expressions for the spinors (Equation 49), we have $\bar{u}^{s'}\gamma^5 u^s \to 0$ in the non-relativistic limit. To find the non-relativistic amplitude, we must therefore go to next to leading order. One can easily check that $\bar{u}^{s'}(\vec{p}')\gamma^5 u^s(\vec{p}) \to m\xi^{s'T}(\vec{p}-\vec{p}')\cdot\vec{\sigma}\xi^s$. So, in the non-relativistic limit, the leading order amplitude arising from pseudoscalar exchange is given by a spin-spin coupling (Figure 3, left)



$$\to +im(-i\lambda)^2 \frac{[\xi^{s'T}(\vec{p}-\vec{p}')\cdot\vec{\sigma}\xi^s][\xi^{r'T}(\vec{p}-\vec{p}')\cdot\vec{\sigma}\xi^r]}{(\vec{p}-\vec{p}')^2+\mu^2}. \tag{54}$$

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