Relative Motion

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1. Inertial frames

Newton's first law of dynamics states that **"If no net force acts on a body it will move in a straight line in constant velocity, or will stay at rest if initially at rest"**. This law can be viewed as a specific example of Newton's second law, $\vec{F} = m\vec{a}$, where $\vec{F} = 0$.

Does this law always holds ?

Consider a ball moving in space at a constant velocity, $\vec{v}$, as viewed by an observer at rest. The ball thus have no forces acting upon it, $\vec{F} = 0$. Consider now a second observer. Let us assume that this second observer moves at constant velocity, $\vec{u}$ with respect to the first observer. In this case, *as viewed by the second observer*, the ball has velocity $\vec{v} - \vec{u}$. Namely, the second observer sees the ball moving at a speed which is different than the speed that would be claimed by the first observer. Nonetheless, this speed is still constant, thus Newton's first law still holds.

Now, let us assume a situation in which the second observer is *accelerating* relative to the first observer. The second observer now sees the ball **decelerating**, although **no force is acting upon it !**. Thus, Newton's first law (and second law as well) seems to be invalid in this case.

Definition. **An *Inertial* reference frame is a frame in which Newton's first law of motion holds**. Similarly, Newton's second law as well as the general conservation principles, of momentum, angular momentum and energy, hold only in inertial frames.

2. Galilean transformation

Two observers will unavoidably watch the same *event* from different locations. Each event is described by its space-time coordinates: t, x, y, z or $t, \vec{x}$, or t, r, θ, ϕ, etc. Often, we are interested in the following question: suppose the first observer tells us the coordinate

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of a given event. What coordinates of the same event will the second observer claim ? The procedure of moving from the first to the second observer's coordinate system is called **coordinate transformation**.

For simplicity, let us work in *Cartesian* coordinate systems. Let us assume that the second observer is moving at velocity $\vec{v}$ in the x-axis with respect to the first observer, as seen by this (first) observer. Moreover, we assume that at time $t = t' = 0$, the origin of the two coordinate systems coincide. From here on, the coordinates of the first observer will be unprimed (t, x, y, z), while those of the second observer will be primed (t', x', y', z').

Consider an event that takes place at position & time t, x, y, z as viewed by the first observer. The second observer will claim that this same event happened at

$$\begin{aligned} t' &= t; \\ x' &= x - vt; \\ y' &= y; \\ z' &= z; \end{aligned} \tag{1}$$

This transformation of coordinates is called **Galilean transformation**, after Galileo Galilee.

Equation 1 is easily generalized to the case where the second observer moves at any particular direction $\vec{x}$ with any particular (yet, constant !) velocity $\vec{v}$, to read

$$\begin{aligned} t' &= t; \\ \vec{x}' &= \vec{x} - \vec{v}t. \end{aligned} \tag{2}$$

The transformation laws of *velocities* of the objects considered are easily derived from equation 1, remembering that the velocity of the second observer, v is constant:

$$\begin{aligned} \frac{dt'}{dt} &= 1; \\ \frac{dx'}{dt'} &= \frac{d(x-vt)}{dt'} = \frac{d(x-vt)}{dt} = \frac{dx}{dt} - v; \\ \frac{dy'}{dt'} &= \frac{dy}{dt}; \\ \frac{dz'}{dt'} &= \frac{dz}{dt}. \end{aligned} \tag{3}$$

Equation 3 can be written in a vectorial form,

$$\vec{u}' = \vec{u} - \vec{v}, \tag{4}$$

where $\vec{u} = \frac{d\vec{x}}{dt}$ and $\vec{u}' = \frac{d\vec{x}'}{dt'}$ are the velocities of the object, as viewed by the first and second observers, respectively.

Equations 3, 4 are known as **Galilean velocity transformation equations**.

Examples.

1. A stone is thrown from a moving train.
2. A sound wave propagating in wind.

Using equations 3, 4, we can derive the transformation laws of accelerations:

$$\begin{aligned} \frac{d^2x'}{dt'^2} &= \frac{d^2x}{dt^2} - \frac{dv}{dt} = \frac{d^2x}{dt^2}; \\ \frac{d^2y'}{dt'^2} &= \frac{d^2y}{dt^2}; \\ \frac{d^2z'}{dt'^2} &= \frac{d^2z}{dt^2}; \end{aligned} \tag{5}$$

This can also be written as

$$\vec{a}' = \vec{a} \tag{6}$$

We thus find that **acceleration of objects are unchanged (conserved) in Galilean transformation**. This is true when transforming from one frame to a frame which moves at constant velocity with respect to the first one. Thus, if Newton's laws hold in the first frame, they will hold in the second frame as well.

Applications:

(1) Given an inertial frame, we can construct any number of inertial frames, each of which moves at a constant velocity relative to the first frame.
(2) Conversely, a frame is inertial only if it has no acceleration relative to other inertial frames.
(3) While two observers located in different inertial frames will not necessarily agree on the position or velocity of an object (system), they will agree on the equation that describes the behavior of the physical system (such as Newton's second law, $\vec{F} = m\vec{a}$); This equation applies to **all** inertial frames.

3. The Center of Mass (CM) frame

One example of Galilean transformation is transformation from the **lab** frame to the **center of mass (CM)** frame. If the lab frame is inertial, so does the CM frame.

Apparently, treatment of many problems in physics is much simpler when working in the CM frame. For example, a system composed of a single particle: the CM frame is the frame in which the particle is at rest; thus, its kinetic energy is zero in this frame.

Let us consider two-body system. The velocity of the CM frame (as viewed in the lab frame) is:

$$\vec{V}_{\rm CM} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2}{m_1 + m_2} = \frac{\vec{P}}{m_1 + m_2}, \tag{7}$$

where $\vec{P} \equiv m_1\vec{u}_1 + m_2\vec{u}_2$ is the total momentum of the system (in the lab frame).

A particle's velocity in the CM frame is obtained by Galilean transformation:

$$\vec{u}_i' = \vec{u}_i - \vec{V}_{\rm CM}. \tag{8}$$

And the velocity of the CM frame, as viewed by an observer in the CM frame (namely, relative its own frame) is obviously zero:

$$\vec{V}_{\rm CM}' = \frac{m_1\vec{u}_1' + m_2\vec{u}_2'}{m_1 + m_2} = \frac{\vec{P}'}{m_1 + m_2} = 0. \tag{9}$$

Thus, the total momentum is the CM frame, $\vec{P}' = \vec{p}_1' + \vec{p}_2' = 0$, or

$$\vec{p}_1' = -\vec{p}_2' \tag{10}$$

Thus, in the CM frame, the particles are moving either towards or away from one another *along the same straight line*, as momentum is conserved.

Example: elastic collision

Let us look at an elastic collision in the CM frame (see Figure 1). Let us denote by $\vec{q}_1', \vec{q}_2'$ the momentum of the colliding particles after the collision.

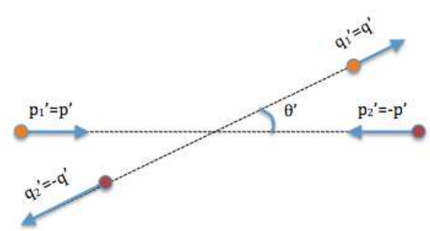

Fig. 1.— An elastic collision in the CM frame.

Conservation of momentum and energy give:

$$\begin{aligned} \vec{p}_1' + \vec{p}_2' &= \vec{q}_1' + \vec{q}_2' = 0 \\ \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2} &= \frac{q_1'^2}{2m_1} + \frac{q_2'^2}{2m_2}. \end{aligned} \tag{11}$$

Let us write $\vec{p}' \equiv \vec{p}_1'$, $\vec{q}' \equiv \vec{q}_1'$. The energy conservation equation is written as:

$$\begin{aligned} \frac{p'^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) &= \frac{q'^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \\ \frac{p'^2}{2m_r} &= \frac{q'^2}{2m_r}, \end{aligned} \tag{12}$$

where $m_r = (m_1^{-1} + m_2^{-1})^{-1}$ is the reduced mass. We thus obtain, $|\vec{p}'| = |\vec{q}'|$.

We are not able to fully solve the problem without specifying the collision angle, θ, but we do obtain that the *magnitude* of the momenta after the collision is fully specified.

If, in the lab frame, one particle is at rest while the second moves at velocity $\vec{v}$ (assuming $m_1 = m_2$), then $\vec{V}_c = \vec{v}/2$, and in the CM frame $\vec{v}_1' = \vec{v}/2$, $\vec{v}_2' = -\vec{v}/2$.

Example 2: inelastic collision.

In a total inelastic collision, the colliding particles merge, forming a single particle of mass $m_1 + m_2$. This particle must be at rest in the CM frame, due to momentum conservation. As such, its total kinetic energy in zero, hence **in inelastic collision, the energy loss is maximal.**

Momentum conservation: $\vec{p}_1' + \vec{p}_2' = 0$.

Energy change:

$$\Delta E' = 0 - \frac{{p_1'}^2}{2m_1} - \frac{{p_2'}^2}{2m_2} = -\frac{p'^2}{2m_r} \tag{13}$$

It is a straightforward algebraic exercise to show that a similar result would be obtained in the lab frame, only with much more effort.

3.1. Coefficient of restitution

We have seen that in a complete elastic collision, the magnitude of the outgoing momenta is similar to the incoming one, in the CM frame: $|\vec{p}'| = |\vec{q}'|$. In the total inelastic collision, the outgoing momenta is $|\vec{q}'| = 0$. In reality, these are two extremes: one can define the **coefficient of restitution**

$$\varepsilon \equiv \frac{|\vec{q}'|}{|\vec{p}'|}, \tag{14}$$

with $0 \leq \varepsilon \leq 1$; $\varepsilon = 1$ for elastic collision, $\varepsilon = 0$ for total inelastic collision. The energy loss in a collision can be written as

$$\begin{aligned} \Delta E' &= \left(\frac{{q_1'}^2}{2m_1} + \frac{{q_2'}^2}{2m_2}\right) - \left(\frac{{p_1'}^2}{2m_1} + \frac{{p_2'}^2}{2m_2}\right) \\ &= \frac{q'^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - \frac{p'^2}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{q'^2}{2m_r} - \frac{p'^2}{2m_r} \\ &= (\varepsilon^2 - 1)\frac{p'^2}{2m_r} = (\varepsilon^2 - 1)E'_{k,0}, \end{aligned} \tag{15}$$

where $E'_{k,0}$ is the initial kinetic energy of the system, in the CM frame.

Note: Sometimes the coefficient of restitution is defined using

$$\varepsilon \equiv \frac{|\vec{v}_1 - \vec{v}_2|}{|\vec{u}_1 - \vec{u}_2|}, \tag{16}$$

where $\vec{u}_i$ is the incoming particle velocity and $\vec{v}_i$ is the outgoing particle velocity. This definition is equivalent to the definition in Equation 15, since the relative velocities are invariant under Galilean transformation:

$$\vec{u}_1 - \vec{u}_2 = \vec{u}_1' - \vec{u}_2' = \frac{\vec{p}'}{m_1} - \left(-\frac{\vec{p}'}{m_2}\right) = \frac{\vec{p}'}{m_r}, \tag{17}$$

and similarly, $\vec{v}_1 - \vec{v}_2 = \vec{q}'/m_r$, from which eq. 15 directly follows.

4. Non-inertial frame: centrifugal force

A great example of a *non-inertial* frame, is a rotating frame, like a rotating disk, or the earth. Let us assume that the frame A is inertial, and denote its coordinates by $\{x, y\}$, and frame B is rotating with respect to frame A at angular velocity ω. We denote the coordinates of frame B by $\{x', y'\}$.

Think of a rotating disk, with an object attached to the edge of the disk. If the object is to be released at time t, an observer at rest (in frame A) will see the object moving in a straight line at a constant velocity; there are no external forces on the object, and it behaves in accordance to Newton's first law (see Figure 2).

Now consider an observer sitting on the disk (frame B). This observer will see the object moving outward in the radial direction (y'), **without any external force acting on it**. Thus, Newton's first law is violated in this frame. The reason is that the observer in B feels a **centripetal acceleration**, $\omega^2 R$, directed inward - towards the center of rotation. The frame B is accelerating with respect to the inertial frame A, hence it cannot be inertial.

4.1. Centrifugal force

As demonstrated above, to an observer located on a rotating disk (e.g., us), an object released would look as if it starts accelerating along the y' axis (away from the center), although no force is acting upon it.

A rotating frame can **appear** to be inertial, by **artificially adding a fictitious outward force** of magnitude $m\omega^2 r$ along the y'-axis. With this force included, an object

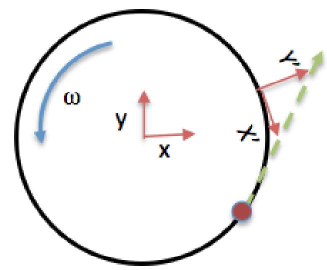

Fig. 2.— An object released from a surface of a rotating sphere. In the inertial frame A, the object seems to move in a straight line, according to Newton's first law. As viewed from a rotating frame B, the object moves in the y' direction, as if it is acted by a force - the *centrifugal force*.

released does obey Newton's laws, as viewed in the frame B. However, this outward force, called **centrifugal force** is not a real force; it is simply a convenient way of making the non-inertial frame B appear inertial, and Newton's laws be applied to this non-inertial frame.

4.2. Earth as a rotating frame

Being on the surface of earth, we are on the surface of a rotating sphere. Thus, at latitude λ, we are at distance $R\cos\lambda$ from the rotation axis, where R is the radius of the earth. Thus, being fixed to the earth's surface, we attach an outward centrifugal force F_c to any moving object. The magnitude of this centrifugal force is

$$F_c = m\omega^2 r = m\omega^2 R\cos\lambda \tag{18}$$

Thus, at the poles: $\lambda = 90°$, and $F_C = 0$, while at the equator $\lambda = 0°$ and $F_C = m\omega^2 R$.

We can introduce this centrifugal force into practical calculations by allowing the gravitational acceleration $\vec{g}$ to vary as a function of latitude. Thus, $\vec{g}$ is replaced by $\vec{g}^* = \vec{g} - \vec{F}_c/m$.

Note that the vectorial nature of the acceleration implies that in general, $\vec{g}^*$ is not directed towards the earth's center (apart from the equator and the poles).

Practically, the difference in the gravitational acceleration at the equator and the poles is small,

$$g^*_{poles} - g^*_{equator} = \omega^2 R = 0.034\ \mathrm{m\,s^{-2}} \tag{19}$$

This can often be ignored, apart when exact calculations are needed. Also note the variation in g^* due to the centrifugal force is comparable to the variation due to the non-sphericity of earth, caused by centrifugal force acted on the liquid surface of earth when it was still molten.

5. Motion in a rotating frame: the Coriolis force

So far, we looked at **static** objects with fixed position with respect to the rotating frame. We saw that, as viewed by an observer **fixed to the rotating frame**, these objects look as if a force is acting upon them - the **centrifugal force**. Let us now broaden the discussion to objects that **move** (in the rotating frame).

Consider an object moving in a straight line (see Figure 3). As viewed from an observer located at the edge of a rotating disk, the object follows a curved path; If the observer is unaware that his frame is rotating, he will ascribe the curvature to an apparent force, which acts perpendicular to the direction of motion. This force is known as **Coriolis force**.

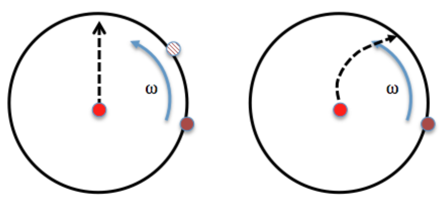

Fig. 3.— Left. A moving object (red ball) moving in a straight line. as viewed by an inertial observer. In this frame, a second observer located on a rotating disk, rotates. Right. The trajectory of the same object, as viewed by an observer located on the rotating disk. The object moves in a curved path.

Preliminary mathematical treatment. Consider an arbitrary vector $\vec{A}'$, **fixed to a rotating frame**, which we denote by O' (to discriminate from the inertial frame, which we denote O). The frame rotates at constant angular velocity $\vec{\omega}$. As viewed by an inertial observer, the vector $\vec{A}'$ changes with time,

$$\frac{d\vec{A}'}{dt} = \vec{\omega} \times \vec{A}' \tag{20}$$

(See Figure 4). As the vector is fixed (in the rotating frame), this change in time is entirely due to the frame rotation. We further note that the rotation of the frame does not change the magnitude of $\vec{A}'$, but only its **direction**. Thus, writing $\vec{A}' = |\vec{A}'|\hat{a}'$, one gets

$$\frac{d\hat{a}'}{dt} = \vec{\omega} \times \hat{a}' \tag{21}$$

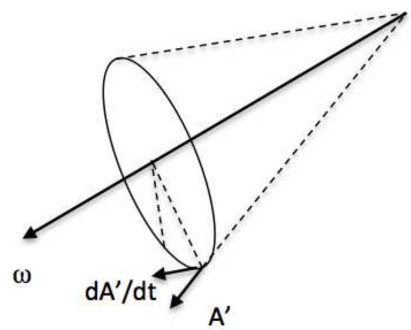

Fig. 4.— A change of a constant vector attached to a rotating frame.

Let us now assume a situation in which the vector $\vec{A}'$ is changed (displaced) during a given time dt', as viewed in the rotating frame. We denote this change by $d'\vec{A}'/dt'$. Note that we use the notation d'/dt' to represent derivative with respect to time as viewed *in the rotation frame*, as opposed to d/dt which represent time derivative in the inertial frame, as these two quantities are generally different. The change in $\vec{A}'$, **as viewed by an inertial observer at** O, is given by

$$\begin{aligned} \frac{d\vec{A}'}{dt} &= \frac{d(|\vec{A}'|\hat{a}')}{dt} \\ &= \frac{d|\vec{A}'|}{dt}\hat{a}' + |\vec{A}'|\frac{d\hat{a}'}{dt} \\ &= \frac{d'|\vec{A}'|}{dt'}\hat{a}' + |\vec{A}'|\left(\frac{d'\hat{a}'}{dt'} + \vec{\omega} \times \hat{a}'\right). \end{aligned} \tag{22}$$

In writing the last line in Equation 22, we used (1) the fact that the magnitude $|\vec{A}'|$ is a **scalar**, and hence invariant under rotations, and (2) Equation 21 that determines the rate of change of $\hat{a}'$ as viewed by an inertial observer.

From Equation 22 one immediately finds

$$\frac{d\vec{A}'}{dt} = \frac{d'\vec{A}'}{dt'} + \vec{\omega} \times \vec{A}' \tag{23}$$

Relating the equation of motion between the two frames. Let us assume that the displacement between the origin of the rotating frame, O' and the origin of the inertial frame O is fixed, and is equal to $\vec{R}$. In this case, consider a general displacement vector $\vec{r}'$ in frame O'. As seen in the inertial frame O, the displacement is $\vec{r} = \vec{R} + \vec{r'}$. If $\vec{R}$ is constant, then $d\vec{r}/dt = d\vec{r'}/dt$.

Under this condition, if $\vec{A'}$ represent a **displacement vector**, $\vec{r'}$, Equation 23 is written as

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r'}}{dt} = \frac{d'\vec{r'}}{dt'} + \vec{\omega} \times \vec{r'}. \tag{24}$$

Similarly, we can apply Equation 23 to the velocity (in the inertial frame), $d\vec{r}/dt$, to write

$$\frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d'}{dt'}\left(\frac{d\vec{r}}{dt}\right) + \vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right). \tag{25}$$

Substituting Equation 24, one finds

$$\begin{aligned} \frac{d^2\vec{r}}{dt^2} &= \frac{d'}{dt'}\left[\frac{d'\vec{r'}}{dt'} + (\vec{\omega} \times \vec{r'})\right] + \vec{\omega} \times \left[\frac{d'\vec{r'}}{dt'} + (\vec{\omega} \times \vec{r'})\right] \\ &= \frac{d'^2\vec{r'}}{dt'^2} + \left(\frac{d'\vec{\omega}}{dt'} \times \vec{r'}\right) + 2\left(\vec{\omega} \times \frac{d'\vec{r'}}{dt'}\right) + \vec{\omega} \times (\vec{\omega} \times \vec{r'}). \end{aligned} \tag{26}$$

Under the assumption that the angular velocity, $\vec{\omega}$ is constant, $d'\omega/dt' = 0$, and the second term in Equation 26 vanishes. Recall that the vector $\vec{r}$ was defined in the rotating frame. Since $d^2\vec{r}/dt = \vec{a}$ is the acceleration, Equation 26 can be written as

$$\vec{a} = \vec{a'} + 2(\vec{\omega} \times \vec{v'}) + \vec{\omega} \times (\vec{\omega} \times \vec{r'}). \tag{27}$$

Equation 27 connects the **acceleration** of a moving object as viewed in an inertial and rotating (non-inertial) frames. We can multiply Equation 27 by m, the mass of the object, to obtain Equation that connects the forces:

$$\begin{aligned} \vec{F} &= \vec{F'} + 2m(\vec{\omega} \times \vec{v'}) + m\vec{\omega} \times (\vec{\omega} \times \vec{r'}); \\ \vec{F'} &= \vec{F} - 2m(\vec{\omega} \times \vec{v'}) - m\vec{\omega} \times (\vec{\omega} \times \vec{r'}). \end{aligned} \tag{28}$$

Note the following:

(1) When transferring from an inertial frame to a rotating frame, a mass appears to experience a change in acceleration, hence forces acting upon it. This confirms the statement that a **rotating frame cannot be inertial.**

(2) The last term in Equation 28, $-m\vec{\omega} \times (\vec{\omega} \times \vec{r'})$ is a vector of magnitude $m\omega^2 r'$ pointing at direction $+\hat{r}'$. This term thus represents the **centrifugal force**.

(3) The second term in Equation 28, $-2m(\vec{\omega} \times \vec{v'})$ appears only when the mass is moving at $\vec{v'} \neq 0$ in the rotating frame. This term is called **Coriolis force** after Gaspard de Coriolis (1835). Similar to centrifugal force, this is a fictitious force, introduced in order to enable motion of bodies, as viewed by an observer sitting in a rotating frame, to be described by Newton's second law of motion.

Examples. The most practical examples of the Coriolis force relates to the rotation of the earth, at angular velocity $\vec{\omega} = 2\pi/T = 7.3 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$, where $T = 24$ h.

1. An object moves at $30\,\mathrm{m\,s^{-1}}$ eastward along the equator. The acceleration due to Coriolis force, $2\vec{\omega} \times \vec{v} = 0.004\,\mathrm{m\,s^{-2}}$ is very small compared to $\vec{g}$.

2. Coriolis force becomes important when treating long-range motions along the earth. In the *northern* hemisphere, the force acts to deflect a body moving north towards the east (see Figure 5). As a consequence, in the northern hemispheres, Hurricanes will **always** rotate counter-clock wise (see Figure 6).

3. As the equatorial region gets more sunlight, its air is warmer, causing winds from the north and south towards the equator. These winds are always deflected towards **west**, and are called *trade winds*.

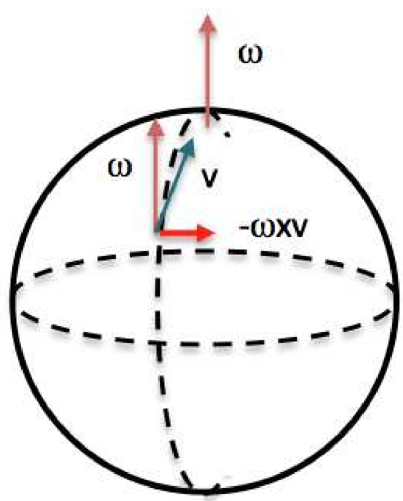

Fig. 5.— Coriolis force deflects objects that move to the north in the northern hemisphere towards the east.

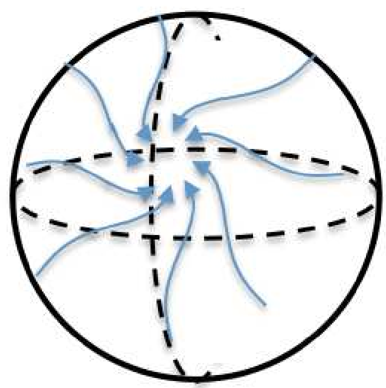

Fig. 6.— Hurricane winds go towards low pressure. Due to Coriolis force, in the northern hemisphere the winds will rotate counter clockwise.

6. The Foucault pendulum

The Foucault pendulum (after Léon Foucault, introduced in 1851) is a simple device that demonstrate the rotation of the earth. This is a simple pendulum, free to swing in the vertical plane. Due to Coriolis force, the actual plane of swing appears to rotate relative to earth.

Consider the pendulum swinging in the east-west direction, in the northern hemisphere. Due to Coriolis force, its path is continuously deflected to the right. Thus, the plane of oscillations rotates clockwise in the northern hemisphere (and counter-clockwise in the southern hemisphere. See Figure 7).

The period of this motion, known as *precession* is $T = 2\pi/(\omega \sin\lambda)$, where λ is the latitude, and $2\pi/\omega$ is the rotation period of the earth (=1 day). Thus, at the pole the plane of oscillations will precess once in a day, while at the equator it is infinity - there is no precession.

The apparent precession of the pendulum is a clear, independent evidence that we are situated on a rotating reference frame. It enables us to determine that the earth is rotating (and the period of rotation), even if we could not observe objects outside of earth.

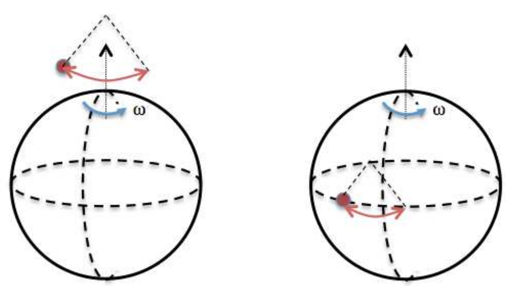

Fig. 7.— Foucault pendulum is a simple pendulum that is free to swing in the vertical plane. In the north pole (left), due to the rotation of the earth, the plane of oscillation of the pendulum rotates clockwise, as viewed by an observer on earth This motion is called *precession*. At the equator (right), no precession is seen.