Rigid body dynamics

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1. Rigid bodies

A **rigid body** is defined as a system of many particles, in which **the distance between each pair of particles is kept fixed.**

We have seen earlier that if an external force and moment are acting upon the body, its momentum and angular momentum change as

$$\begin{array}{rcl} \frac{d\vec{P}}{dt} & = & \vec{F}_{ext} \\ \frac{d\vec{L}}{dt} & = & \vec{M}_{ext}. \end{array} \tag{1}$$

Here, $\vec{P}$ is the total linear momentum, corresponding to the center-of mass (CM) velocity $\vec{V}_c$, thus $\mathcal{M} d\vec{V}_c/dt = \vec{F}_{ext}$, where $\mathcal{M}$ is the mass of the body.

As a specific example, consider a body of mass $\mathcal{M}$ having arbitrary shape, which is located near the earth's surface. The body can be thought of as comprising a large number of small masses, $m_1, m_2, ...m_i...$, all being subject to the same gravitational force, $m_i\vec{g}$. Assuming that gravity is the only force in the system, one gets $\vec{F}_{ext} = \sum_i m_i \vec{g} = \mathcal{M}\vec{g}$. Similarly, the net external moment is $\vec{M}_{ext} = \sum_i \vec{r}_i \times m_i \vec{g}$.

If C is the center of mass, then its displacement relative to the coordinate origin, O, is defined by $\vec{R}_c = \sum_i m_i \vec{r}_i / \sum_i m_i$. The displacement of a single mass element can be written as $\vec{r}_i = \vec{R}_c + \vec{r}'_i$, where $\vec{r}'_i$ is the displacement of m_i relative to C. One can thus write the net moment,

$$\begin{array}{rcl} \vec{M}_{ext} & = & \sum_i \vec{r}_i \times m_i \vec{g} = \sum_i (\vec{R}_c + \vec{r}'_i) \times m_i \vec{g} \\ & = & \sum_i \vec{R}_c \times m_i \vec{g} + \sum_i \vec{r}'_i \times m_i \vec{g} \\ & = & \vec{R}_c \times \left(\sum_i m_i\right) \vec{g} + \left(\sum_i m_i \vec{r}'_i\right) \times \vec{g} \\ & = & \vec{R}_c \times \mathcal{M}\vec{g} + \left(\sum_i m_i \vec{r}'_i\right) \times \vec{g} \end{array} \tag{2}$$

By definition of the center of mass, $\sum_i m_i \vec{r}'_i = 0$, and we are thus left with

$$\vec{M}_{ext} = \vec{R}_c \times \mathcal{M}\vec{g}. \tag{3}$$

We thus conclude that **the moment acting on a rigid body is similar to the moment that acts on a point-body of a similar mass, located at the center-of-mass**.

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2. Statics

For a rigid body at rest, namely $\vec{P} = 0, \vec{L} = 0$, two conditions are met: **the sum of external forces and the sum of external moments -** *relative to any origin* **are zero,** $\vec{F}_{ext} = 0; \vec{M}_{ext} = 0$. Note that these same conditions are also satisfied in the case where the CM is moving at constant speed ($\vec{P} = const$), and/or the body is rotating at constant angular momentum ($\vec{L} = const$).

These conditions are sufficient in analyzing any system.

Examples.

1. Consider a rigid beam at rest, hanged from a point O located at distance d from its CM (see Figure 1). On the beam, there are 3 masses hanged, such that the beam remains horizontal. The net forces on the beam:

$$\vec{F}_1 + \vec{F}_2 + m\vec{g} + \vec{F}_3 + \vec{F}_R = 0, \tag{4}$$

where $\vec{F}_R$ is the reaction exerted upward on the beam, and $m\vec{g}$ is the weight of the beam.

Taking moments relative to O [point chosen arbitrarily],

$$(\vec{d}_1 \times \vec{F}_1) + (\vec{d}_2 \times \vec{F}_2) + (\vec{d}_3 \times \vec{F}_3) + (\vec{d} \times m\vec{g}) + (0 \times \vec{F}_R) = 0. \tag{5}$$

Since the moments extracted by forces $1, 2$ are in the opposite direction to the moment extracted by the force 3 and by the rod itself, equation 5 can be written as

$$|\vec{F}_1|d_1 + |\vec{F}_2|d_2 - |\vec{F}_3|d_3 - m|\vec{g}|d = 0 \tag{6}$$

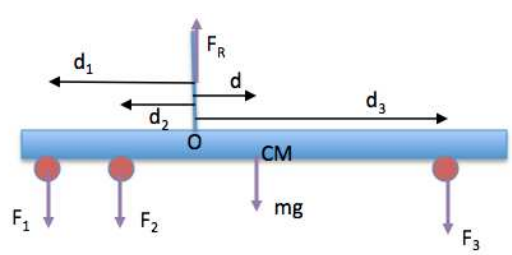

Fig. 1.— A rigid beam hanged from point O, with masses attached to it. In a static situation, the new forces and net moments acting on the bean is 0

2. Consider a uniform ladder of length l and mass m at rest against a vertical wall (see Figure 2). The ladder is making an angle θ with respect to the horizontal. We neglect the

friction between the ladder and the wall (but not the friction between the ladder and the floor !).

Equilibrate the forces: in the x, y directions

$$\begin{aligned} |R| - |f_R| &= 0, \\ |N| - m|g| &= 0. \end{aligned} \tag{7}$$

Moments of forces relative to point B:

$$m|g|(l/2)\cos\theta - |R| l \sin\theta = 0, \tag{8}$$

with the solution $|N| = m|g|$ and $|R| = |f_R| = (1/2)m|g|\cot\theta$.

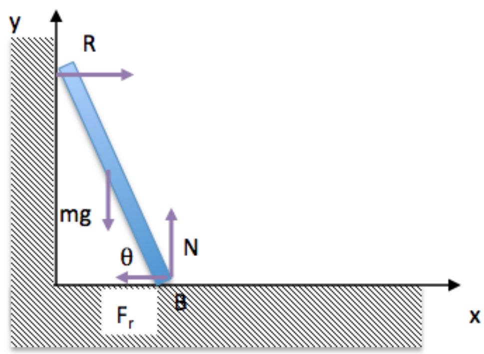

Fig. 2.— A uniform ladder hanging against a wall.

3. Torque

Consider an arbitrary rigid body which is acted upon by several forces, $\vec{F}_1, \vec{F}_2, ...\vec{F}_j$. Denote by $\vec{r}_i$ the displacement of a unit mass m_i relative to some arbitrary origin, O, and by $\vec{r}_i'$ its displacement relative to a different origin, O'. Denote by $\vec{R}$ the displacement between O and O'. Then $\vec{r}_i = \vec{R} + \vec{r}_i'$.

The total moment experienced by the rigid body due to the forces relative to O is $\vec{M}_{ext} = \sum_i \vec{r}_i \times \vec{F}_i$, and relative to O' is $\vec{M}'_{ext} = \sum_i \vec{r}_i' \times \vec{F}_i$. Thus,

$$\begin{aligned} \vec{M}_{ext} &= \textstyle\sum_i \vec{r}_i \times \vec{F}_i = \sum_i (\vec{r}_i' + \vec{R}) \times \vec{F}_i \\ &= \textstyle\sum_i \vec{r}_i' \times \vec{F}_i + \sum_i \vec{R} \times \vec{F}_i \\ &= \vec{M}'_{ext} + \left(\vec{R} \times \sum_i \vec{F}_i\right). \end{aligned} \tag{9}$$

Equation 9 has an important consequence: **if the sum of external forces is zero, $\sum_j \vec{F}_j = 0$, then the net moment experienced by the body is independent on the origin of the coordinates,** $\vec{M}_{ext} = \vec{M}'_{ext}$. In this case, the moment $\vec{M}_{ext}$ is called **torque**, and is often denoted by $\vec{T}$.

Example.

Consider two forces acting on a body, with similar magnitude and opposite directions, so that the net force is $\vec{F}_1 + \vec{F}_2 = 0$. The torque exerted on the body is

$$\begin{aligned} \vec{T} &= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_1, \\ |\vec{T}| &= |\vec{F}|d, \end{aligned} \tag{10}$$

where d is the perpendicular distance between the lines of action of the forces.

4. Dynamics of a rigid body

Consider a rigid body, with an external force $\vec{F}_{ext}$ acting upon it. Denote by C the position of the CM of the body, relative to arbitrary origin O. Applying equation 9, we find that the total moment (relative to O) is

$$\vec{M}_{ext} = \vec{M}'_{ext} + \vec{R}_c \times \vec{F}_{ext}, \tag{11}$$

where $\vec{M}'_{ext} = \sum_i \vec{r}'_i \times \vec{F}_{ext}$ is the moment relative to C. Using Equation 1, we find

$$\begin{aligned} \tfrac{d\vec{P}}{dt} &= \vec{F}_{ext} = \mathcal{M}\tfrac{d\vec{V}_c}{dt} \\ \tfrac{d\vec{L}}{dt} &= \vec{M}_{ext} = \vec{M}'_{ext} + \vec{R}_c \times \vec{F}_{ext} = \vec{M}'_{ext} + \vec{R}_c \times \mathcal{M}\tfrac{d\vec{V}_c}{dt} \end{aligned} \tag{12}$$

We can interpret eq. 12 as follows: the motion of a body that is subject to external force can be viewed as the sum of: (1) the motion of a point body of mass $\mathcal{M}$ located at the CM due to the force $\vec{F}_{ext}$; and (2) rotation of the body around the CM due to the influence of the net moment $\vec{M}'_{ext}$.

If there is no net force, $\vec{F}_{ext} = 0$, $d\vec{V}_c/dt = 0$ and a rigid body will experience only torque, $\vec{M}_{ext} = \vec{M}'_{ext} = \vec{T}_{ext}$. Equation 12 becomes

$$\frac{d\vec{L}}{dt} = \vec{T}_{ext} \tag{13}$$

Example. If we want to rotate a wheel at a maximum rate (=change its angular momentum), we need to apply maximum **torque**. We need to apply the force perpendicular to the wheel surface.

5. Rotation of a rigid body around a fixed axis: moment of inertia

Consider a rigid body rotating around a fixed axis. Every point moves in a circle, with angular velocity $\vec{\omega}$ about the axis of rotation. Assume that the point mass (m_i) is located at radius vector $\vec{r}_i$ from the rotation axis. The instantaneous (translational) velocity of the particle in the direction perpendicular to $\vec{r}_i$ (tangential to the circle) is $\vec{v}_i = \vec{\omega} \times \vec{r}_i$. Thus, the angular momentum of the particle is $\vec{L}_i = m_i \vec{v}_i \times \vec{r}_i$, directed towards the rotation axis.

The magnitude of the total angular momentum of the rigid body is thus

$$|L| = \sum_i m_i |\vec{v}_i||\vec{r}_i| = \sum_i |\omega| m_i \vec{r}_i^2 = |\omega| \sum_i m_i \vec{r}_i^2. \tag{14}$$

This result can be compared to the definition of the angular momentum, $\vec{P} = m\vec{v} = \vec{v}\sum_i m_i$; Thus, in analogue to linear motion, in rotational motion $\vec{\omega}$ replaces $\vec{v}$ and $\sum_i m_i \vec{r}_i^2$ replaces m. This quantity is called **moment of inertia** of a body, **around a fixed axis,** A,

$$I_A \equiv \sum_i m_i r_i^2 \tag{15}$$

The angular momentum, Equation 14 is thus

$$\vec{L} = \vec{\omega} I_A \tag{16}$$

The moment of inertia depends on (1) the mass distribution of the body; and (2) the rotation axis. Note that similarly to linear momentum, when the (scalar) mass represents the body reaction to an applied force, the (scalar) moment of inertia represents the body reaction to an applied torque.

6. Calculation of the moment of inertia: examples

6.1. Solid cylinder

Consider a uniform cylinder of radius R, mass M and length L, rotating around its axis of symmetry. The cylinder mass can be written as $M = \pi R^2 L \rho$, where ρ is its mass density. We can think of the cylinder as being composed of many sub-cylinders, each of radius r, $0 \leq r \leq R$ and width Δr. The mass element in each sub-cylinder is thus $\Delta M = 2\pi r \Delta r L \rho$. The moment of inertia is

$$\begin{aligned} I_x &= \int\int\int_V r^2 dM = \int_0^R 2\pi r^3 dr L\rho \\ &= 2\pi L\rho \int_0^R r^3 dr = \tfrac{\pi}{2} L\rho R^4 \\ &= (\pi L\rho R^2)\tfrac{R^2}{2} = \tfrac{1}{2} M R^2 \end{aligned} \tag{17}$$

6.2. Ball (uniform sphere) rotating around its diameter

Obviously, it is natural to use *spherical coordinates*, r, θ, ϕ. A volume element of the ball is $dV = dr(rd\theta)(r\sin\theta d\phi) = r^2 \sin\theta dr d\theta d\phi$. The mass of a volume element is $dm = \rho dV$, and its distance from the rotation (z) axis is $r\sin\theta$. The moment of inertia is thus

$$\begin{aligned} I_z &= \int\int\int_V (r\sin\theta)^2 dm \\ &= \int\int\int_V (r\sin\theta)^2 \rho r^2 \sin\theta dr d\theta d\phi \\ &= \rho \int_0^R r^4 dr \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} d\phi \\ &= \rho \tfrac{R^5}{5} \times \left(-\int_0^\pi \sin^2\theta d\cos\theta\right) \times 2\pi \\ &= \rho \tfrac{2\pi}{5} R^5 \int_{-1}^1 (1-\lambda^2) d\lambda = \tfrac{4}{3}\tfrac{2\pi}{5}\rho R^5 = \tfrac{2}{5} M R^2 \end{aligned} \tag{18}$$

6.3. The parallel axes theorem

Let us assume that I_C, the moment of inertia of a body of mass $\mathcal{M}$ about an axis passing through the center of mass (C) is known. We wish to calculate the moment of inertia about an axis perpendicular to the first axis, yet passing through a different point, A. Denote by l the perpendicular distance between C and A, so that the displacement of a mass element be $\vec{r}_i'$, $\vec{r}_i$ with respect to A and C: $\vec{r}_i' = \vec{r}_i + \vec{l}$.

The moment of inertia about an axis passing through A is thus

$$\begin{aligned} I_A &= \sum_i m_i r_i'^2 \\ &= \sum_i m_i (r_i + l)^2 \\ &= \sum_i m_i {r_i}^2 + 2\sum_i m_i \vec{r}_i \cdot \vec{l} + \sum_i m_i l^2 \\ &= \sum_i m_i {r_i}^2 + 2\left(\sum_i m_i \vec{r}_i\right) \cdot \vec{l} + \mathcal{M} l^2 \\ &= I_C + \mathcal{M} l^2 \end{aligned} \tag{19}$$

Equation 19 is known as **the parallel axes theorem**, and is extremely useful in calculating the moment of inertia about different axes.

7. Conservation of angular momentum of rigid bodies

We have seen that when a torque is applied to a body, its angular momentum changes,

$$\vec{T}_{ext} = \frac{d\vec{L}}{dt} = \frac{d(I_A\vec{\omega})}{dt}. \qquad (20)$$

In the absence of external torque, $\vec{T}_{ext} = 0$, we find that the angular momentum is conserved, $\vec{L} = I_A\vec{\omega} = const$.

If, in this case, the moment of inertia changes, than so will the angular velocity.

Example. A spinning dancer putting his/hers hands outward.

Finally, we note the analogy between Newton's second law for change of linear momentum, $\vec{F} = d\vec{P}/dt = md\vec{V}/dt$, and the equation for change of angular momentum, $\vec{T} = d\vec{L}/dt = Id\vec{\omega}/dt$.

8. Mechanical energy of rotating bodies

As a body rotates, a mass element m_i moves at a circle of radius $\vec{r}_i$ around its rotation axis. The kinetic energy associated with this motion is $K_i = (1/2)m_iv_i^2 = (1/2)m_ir_i^2\omega^2$. Thus, thus total kinetic energy is

$$E_k = \sum_i \frac{1}{2}m_ir_i^2\omega^2 = \frac{1}{2}\omega^2\sum_i m_ir_i^2 = \frac{1}{2}I\omega^2. \qquad (21)$$

This can be written as $E_k = (1/2)L^2/I$.

Note the analogy with the kinetic energy associated with the linear motion, $E_{K,linear} = (1/2)mv^2$.

We can apply the parallel axes theorem, Equation 19, to write the kinetic energy associated with rotation around axis A as a function of the kinetic energy associated with rotation around the center of mass, C,

$$\begin{aligned} E_K(A) &= \tfrac{1}{2}I_A\omega^2 \\ &= \tfrac{1}{2}(I_C + \mathcal{M}l^2)\omega^2 \\ &= \tfrac{1}{2}I_c\omega^2 + \tfrac{1}{2}\mathcal{M}l^2\omega^2 \\ &= \tfrac{1}{2}I_c\omega^2 + \tfrac{1}{2}\mathcal{M}V_C^2 \end{aligned} \qquad (22)$$

Example 1. A rolling cylinder.

Consider a wheel of radius R and mass $\mathcal{M}$ rolling on an inclined plane (see Figure 3). Assume that the cylinder is at rest at point A, and is rolling without sliding. What is its velocity as it reaches point B, which is lower than point A by height h ? Conservation of energy gives

$$\mathcal{M}gh = \frac{1}{2}\mathcal{M}V_c^2 + \frac{1}{2}I\omega^2 \tag{23}$$

The instantaneous velocity V_c of the center of mass is related to the angular velocity ω by $V_c = \omega R$. Moreover, we derived in Section 6.1, Equation 17 the moment of inertia of a cylinder, $I = (1/2)\mathcal{M}R^2$. We thus find

$$\mathcal{M}gh = \frac{1}{2}\mathcal{M}V_c^2 + \frac{1}{4}\mathcal{M}V_c^2 = \frac{3}{4}\mathcal{M}V_c^2, \tag{24}$$

from which we readily obtain that $V_c = \sqrt{(4/3)gh}$. This is **less** than the result that would be obtained if the cylinder was to slide without friction, $V_c = \sqrt{2gh}$, due to the energy that was converted to rotation.

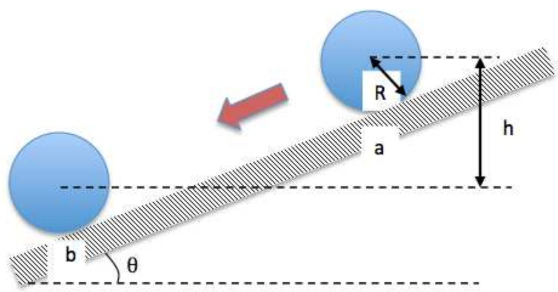

Fig. 3.— A cylinder rolling on an inclined plane.

Example 2. A physical pendulum.

Consider a rigid body that is hanged from a point A, such that it is free to rotate. If the body is displaced from equilibrium, it oscillates back and forth under the influence of a gravitational field (see Figure 4). We are interested in calculating the oscillation period of the body.

Denote by l the perpendicular distance between its center of mass C and A. The point C thus moves in a circle of radius l. The total energy is the sum of the gravitational potential energy, and the kinetic energy associated with the rotation around A.

The potential energy is easily calculated, assuming that all the mass is concentrated on the center of mass, thus $V = mgy$, where y is the displacement of C above its lowest point.

Conservation of energy thus gives

$$E = mgy + \frac{1}{2} I_A \omega^2 = const. \tag{25}$$

Using $y = l(1 - \cos\theta)$ and differentiating with respect to time one gets,

$$mgl \sin\theta \frac{d\theta}{dt} + \frac{1}{2} I_A 2\omega \frac{d\omega}{dt} = 0, \tag{26}$$

and since $\omega = d\theta/dt$,

$$\frac{mgl \sin\theta}{I_A} + \frac{d^2\theta}{dt^2} = 0. \tag{27}$$

For small angles, one can approximate $\sin\theta \approx \theta$, and get

$$\frac{d^2\theta}{dt^2} = -\frac{mgl}{I_A}\theta. \tag{28}$$

This is identical to the oscillation of a simple harmonic oscillator, with period given by

$$T = 2\pi\sqrt{\frac{I_A}{mgl}} \tag{29}$$

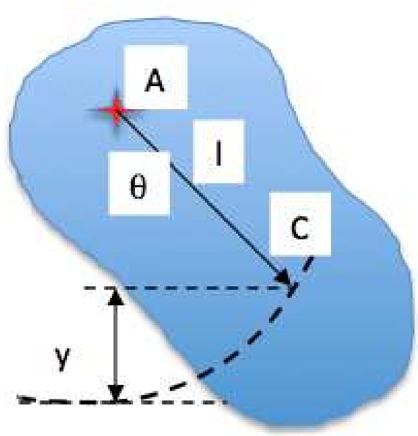

Fig. 4.— A physical pendulum (rigid body) hanged from point A.

9. Summary

The following table summarized the analogy between linear motion and rotational motion.

Linear motion		**rotational motion**	
Displacement	$\vec{r}$	Angular displacement	$\vec{\theta}$
Velocity	$\vec{v} = \frac{d\vec{r}}{dt}$	Angular velocity	$\vec{\omega} = \frac{d\vec{\theta}}{dt}$
mass	m	moment of inertia	$I_A = \sum_i m_i r_i^2$
Force	$\vec{F}$	Torque	$\vec{T} = \vec{r} \times \vec{F}$
momentum	$\vec{P} = m\vec{v}$	angular momentum	$\vec{L} = I_A\vec{\omega} = \vec{r} \times \vec{P}$
Equation of motion	$\vec{F} = \frac{d\vec{P}}{dt} = m\frac{d\vec{v}}{dt}$		$\vec{T} = \frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt}$
conservation laws	$\vec{F} = 0 \rightarrow \vec{P} = const$		$\vec{T} = 0 \rightarrow \vec{L} = const$
Kinetic energy	$K = \frac{1}{2}mv^2$		$K = \frac{1}{2}I_A\omega^2$
Work	$W = \int \vec{F} \cdot d\vec{r}$		$W = \int \vec{T} \cdot d\vec{\theta}$